

## 4.2 Free Modules

November. 20, 2008

**Example 1.**  $\mathbb{R} = \mathcal{M} = \text{Hom } (\mathbb{Z}_2[x], \mathbb{Z}_2[x]).$  Hence  $\mathcal{M} = \mathbb{R}$  is  $\mathbb{R}$  – module and 1 is basis.  
Let  $f_1, f_2 \in \mathcal{M}$  s.t

$$f_1(x^{2i}) = x^i, f_1(x^{2i+1}) = 0, f_2(x^{2i}) = 0 \text{ and } f_2(x^{2i-1}) = x^i \text{ for } i \in \mathbb{N} \cup \{0\}.$$

*Claim:*  $f_1, f_2 \in \mathcal{M}$  are linear independent.

Suppose  $g_1 f_1 + g_2 f_2 = 0$  for  $g_1, g_2 \in \mathbb{R}.$

$$\text{Then } 0 = g_1 f_1 + g_2 f_2(x^{2i}) = g_1 f_1(x^{2i}) = g_1(x^i).$$

$$0 = g_1 f_1 + g_2 f_2(x^{2i-1}) = g_2 f_2(x^{2i-1}) = g_2(x^i) \text{ for } i \in \mathbb{N} \cup \{0\}.$$

Hence  $g_1 = g_2 = 0.$

*Claim:*  $f_1, f_2$  span  $\mathcal{M}.$

Pick any  $g \in \mathcal{M}$ , Pick  $g_1, g_2 \in \mathbb{R}.$

$$\text{s.t } g_1(x^i) = g(x^{2i}) \text{ and } g_2(x^i) = g(x^{2i-1}) \text{ for } i \in \mathbb{N} \cup \{0\}.$$

$$\text{Then } (g_1 f_1 + g_2 f_2)(x^{2i}) = g_1 f_1(x^{2i}) = g_1(x^i) = g(x^{2i}).$$

$$\text{and } (g_1 f_1 + g_2 f_2)(x^{2i-1}) = g_2 f_2(x^{2i-1}) = g_2(x^i) = g(x^{2i-1}).$$

Then  $g = g_1 f_1 + g_2 f_2.$

We have shown  $\mathcal{M} \cong \mathbb{R} \cong \mathbb{R}^2 \cong \mathbb{R}^3 \dots$

**Definition 1.** Let  $\mathcal{M}, \mathcal{N}$  be  $\mathbb{R}$  – modules.

$$\text{Let } \mathcal{M} \oplus \mathcal{N} = \{(m, n) | m \in \mathcal{M}, n \in \mathcal{N}\}$$

and + and scalar multiplication are defined component.

Then  $\mathcal{M} \oplus \mathcal{N}$  is  $\mathbb{R}$  – modules , called the direct sum  $\mathcal{M}$  and  $\mathcal{N}.$

**Note 1.** :  $\mathcal{M} \oplus \mathcal{N} \oplus \mathcal{T}$  can be defined similarly for  $\mathbb{R}$  – modules  $\mathcal{M}, \mathcal{N}, \mathcal{T}.$