4.3 Rejective Module

Definition. A R-module P is projective if

$$\begin{array}{ccccc} & & P & & \\ & h \swarrow & \circlearrowright \downarrow f & & \\ M & \underset{g}{\longrightarrow} & N & \longrightarrow & O \end{array}$$

i.e. Given any exact sequence $M \xrightarrow{g} N \longrightarrow O$ and $f : P \to M$ s.t. the daigram

$$\begin{array}{ccccc} & & P & & \\ & h \swarrow & \circlearrowright \downarrow f & & \\ M & \underset{g}{\longrightarrow} & N & \longrightarrow & O \end{array}$$

is commutative.

Note. 1. *If P is a free module, then $f : P \to N$ is determined by the map of one basis.*

2. *If P is a free R-module with basis $\{e_i\}$, then we define $h(e_i)$ to be an element $m_i \in M$ s.t. $g(m_i) = f(e_i)$ (g is onto). We have $f = gh$.*

3. *A free R-module is projective.* $\square$

Theorem. *Let P be a R-module*
TFAE:

1. *P is projective.*

2. *Any exact sequence*
   $O \longrightarrow L \longrightarrow M \longrightarrow P \longrightarrow O$
   splits.

3. *$L \oplus P$ is a free R-module for some R-module L.*

Example. $R = \mathbb{Z}_6$
$\mathbb{Z}_2$ is not a free R-module since 1 is not a basis.
$\mathbb{Z}_3$ is not a free R-module.
But $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6$ is a free R-module with basis $\{1\}$.
Then $\mathbb{Z}_2, \mathbb{Z}_3$ are projective R-module by Theorem 3.

Proof:

- (1) $\Rightarrow$ (2)

$$\begin{array}{ccccccccc} O & \longrightarrow & L & \longrightarrow & M & \longrightarrow & P & \longrightarrow & O \\ & & & & & h \swarrow \circlearrowright & \downarrow id & & \\ & & & & M & \longrightarrow & P & \longrightarrow & O \end{array}$$

Since P is projective, there exist $h : P \to M$ s.t. the above diagram is commutative. Then we have (2) by previous theorem.

- $(3) \Rightarrow (1)$

$$\begin{array}{ccccc} & & L \oplus P & & \\ & & \downarrow\uparrow & & \\ h' \swarrow & \circlearrowleft & P & & \\ & & \downarrow f & & \\ M & \underset{g}{\longrightarrow} & N & \longrightarrow & O \end{array}$$

$\Rightarrow$ h=hP is what we want.

- $(2) \Rightarrow (3)$
Let$X \subseteq P$ be a generating set of P.
Then $M = \oplus_{x \in X} R$ is a free R-module.
Define $g : M \to P$ canonically.
Then $O \longrightarrow Ker(g) \xrightarrow{id} M \xrightarrow{g} P \longrightarrow$ O.
Then by (2),$M \cong Ker(g) \oplus P$ is a free R-module. $\square$