

4.4 Homomorphism

January 11, 2009

Definition 1 $\text{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^s) = \{f \mid f : \mathbb{R}^n \rightarrow \mathbb{R}^s \text{ is a } \mathbb{R}\text{-module homomorphism}\}$

For $f, g \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^s)$, define $(f+g)u = f(u)+g(u)$, $(cf)(u) = cf(u)$ for $c \in \mathbb{R}$ and $u \in \mathbb{R}^n$. Then $\text{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^s)$ is a \mathbb{R} -module.

$\text{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^s) \otimes_{\mathbb{R}} \text{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^t)$ is isomorphic to $\text{Hom}_{\mathbb{R}}(\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^m, \mathbb{R}^s \otimes_{\mathbb{R}} \mathbb{R}^t)$ by the map $A \otimes_{\mathbb{R}} B \rightarrow \phi(A \otimes_{\mathbb{R}} B)$ where $\phi(A \otimes_{\mathbb{R}} B)(u \otimes_{\mathbb{R}} v) = Au \otimes_{\mathbb{R}} Bv$, $A \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^s)$, $B \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^t)$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$.

For convenience, we will write $A \otimes_{\mathbb{R}} B$ for $\phi(A \otimes_{\mathbb{R}} B)$ or the matrix form of $\phi(A \otimes_{\mathbb{R}} B)$, and \otimes for $\otimes_{\mathbb{R}}$.

Let $\{e_i\}, \{f_i\}, \{e'_i\}, \{f'_i\}$ be the bases of $\mathbb{R}^n, \mathbb{R}^s, \mathbb{R}^m, \mathbb{R}^t$ respectively. Then $A \otimes B(e_i \otimes e'_j) = (Ae_i) \otimes (Be'_j) = (\sum_k a_{ki} f_k) \otimes \sum_u b_{uj} f'_u = \sum_{k,u} a_{ki} b_{uj} (f_k \otimes f'_u)$ for some $a_{ki}, b_{uj} \in \mathbb{R}$.

i.e. $A \otimes B$ has matrix form $\begin{bmatrix} a_{11}B & a_{12}B & \dots \\ a_{21}B & a_{22}B & \dots \\ \vdots & \vdots & \end{bmatrix}$ if the basis of $\mathbb{R}^n \otimes \mathbb{R}^m$ is ordered by $e_1 \otimes e'_1, e_1 \otimes e'_2, \dots, e_2 \otimes e'_1, \dots$, and the basis of $\mathbb{R}^s \otimes \mathbb{R}^t$ is ordered by $f_1 \otimes f'_1, f_1 \otimes f'_2, \dots, f_2 \otimes f'_1, \dots$.

Note

- (1) $B \otimes A = P(A \otimes B)Q$ for some permutation matrices P, Q .
- (2) $(A \otimes B)^t = A^t \otimes B^t$ if $n = s, m = t$.
- (3) $(A \otimes B)(A' \otimes B') = (AA') \otimes (BB')$ for $A' \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^n)$, $B' \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^m)$ if $n = s, m = t$.

Proof of (3)

$$((A \otimes B)(A' \otimes B'))(e_i \otimes e'_j) = A(A'e_i) \otimes B(B'e'_j)$$

$$= (AA')e_i \otimes (BB')e'_j = ((AA') \otimes (BB'))e_i \otimes e'_j \quad \square$$

(4) If $n = s, m = t$, and A, B invertible, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

(5) If $n = s, m = t$, and λ_i is an eigenvalue of A with corresponding eigenvector u_i , η_j is an eigenvalue of B with corresponding eigenvector v_j , then $\lambda_i\eta_j$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $u_i \otimes v_j$.

Proof of (5)

$$A \otimes B(u_i \otimes v_j) = Au_i \otimes Bv_j = (\lambda_i u_i) \otimes (\eta_j v_j) = (\lambda_i \eta_j)(u_i \otimes v_j) \quad \square$$