4.5 Tensor product

R is a commutative ring in this section.

Definition 4.1. Let M, N, O be R-modules. A function $f : M \times N \to O$ is *bilinear* if the maps $f(*, n) : M \to O$ and $f(m, *) : N \to O$ are R-module homomorphisms for each $n \in N$ and $m \in M$.

Note 4.2. *1. $M \times N$ is not a R-module*

2. $M \oplus N$ is a R-module with underline set $M \times N$

3. If $f : M \oplus N \to O$ is a R-module homomorphism, then $f((cm, cn)) = f(c(m, n)) = cf((m, n))$ for $c \in R, m \in M$, and $n \in N$

4. If $f : M \times N \to O$ is bilinear then $f(cm, cn) = cf(m, cn) = c^2 f(m, n)$ for $c \in R, m \in M$, and $n \in N$

Definition 4.3. Let M, N be R-modules. The *tensor product* $M \otimes_R N$ of M and N is a module toghther with a bilinear map $f : M \times N \to M \otimes_R N$ such that for any bilinear map $g : M \times N \to O$ for some R-module O, there exists a R-module homomorphism $h : M \otimes_R N \to O$ with the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[dr, "g"'] & M \otimes_R N \arrow[d, "h"] \arrow[dl, phantom, "\circlearrowright"] \\ & O \end{tikzcd}$$

Theorem 4.4. *$M \otimes_R N$ exists.*

Proof. Set F to be the free module with basis $M \times N$ (i.e., $F = \oplus_{x \in M \times N} R = \sum_{x \in M \times N} rx$. Let K be a R-submodule of F generated by $(m + m', n) - (m, n) - (m', n), (m, n + n') - (m, n) - (m, n'), (cm, n) - c(m, n)$, and $(m, cn) - c(m, n)$ for any $m, m' \in M$, $n, n' \in N$, and $c \in R$. We shall prove $M \otimes_R N = F/K$. Define a map $f : M \times N \to F/K$ by $f(m, n) = (m, n) + K$. Check f is bilinear. $f(m + m', n) = (m + m', n) + K = (m + m', n) - ((m + m', n) - (m, n) - (m', n)) + K = (m, n) + (m', n) + K = f(m, n) + f(m', n)$. Similar for others. Suppose we have

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[dr, "g"'] & F/K \\ & O \end{tikzcd}$$

Define $h : F/K \to O$ by extending the definition $h(f(m, n) + K) = g(m, n)$, clearly $hf = g$. It is routine to check h is a well-define homomorphism. $\square$

Theorem 4.5. *$M \otimes_R N$ is unique.*

Proof. Suppose A, B are 2 modules satisfying the definition of tensor product. Then we have

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[dr, "g"'] & A = F/K \arrow[d, shift right, "h"'] \arrow[dl, phantom, "\circlearrowright"] \\ & B \arrow[u, shift right, "k"'] \end{tikzcd}$$

i.e., $g = hf$ and $f = kg$. Then $g = hf = hkg$ and $f = kg = khf$. Hence $hk = 1_{img(g)}, kh = 1_{img(f)}$. Thus, $h : A \to B$ is a module isomorphism. $\square$

Note 4.6. *We will write* $m \otimes n$ *for* $f(m,n)$ *with* $m \in M, n \in N$.

Example 4.7. $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Q} = ?$
$1 \otimes \frac{b}{a} = 1 \otimes \frac{2b}{2a} = 2\left(1 \otimes \frac{b}{2a}\right) = 2 \otimes \frac{b}{2a} = 0 \cdot 0 \otimes \frac{b}{2a} = 0 \otimes 0$. Hence $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Q} = \{0 \otimes 0\}$.