We assume $R = \mathbb{R}$.

Fact:If $e_1, e_2, ..., e_s$ are linear independent in the R-module M and $n_1, n_2, ..., n_s$ ϵ N such that $e_1 \bigotimes n_1 + e_2 \bigotimes n_2 + ... + e_s \bigotimes n_s = 0$, then $n_1 = n_2 = ...n_s = 0$ (Theorem 5.11).

(If write element in M$\times N$ as a sum of $e_i \bigotimes n_i$, there is no way to reduce the length of the sum.)

Theorem:$\{e_i\}_{i=1}^a$ is a basis of M and $\{f_j\}_{i=1}^b$ is a basis of N $\Rightarrow$ $\{e_i \bigotimes f_j\}$ is a basis of $M \bigotimes_{\mathbb{R}} N$.

Proof: Span: Pick any $m \times n \epsilon M \bigotimes_{\mathbb{R}} N$,

suppose

$$m = \sum_{i=0}^{a} c_i e_i \text{ and } n = \sum_{j=1}^{b} d_j f_j,$$

then

$$m \bigotimes n = (\sum c_i e_i) \bigotimes (\sum d_j f_j) = \sum_i \sum_j c_i d_j (e_i \bigotimes f_j),$$

linear independent:

Suppose

$$\sum_{i,j} c_i d_j (e_i \bigotimes f_j) = 0,$$

then

$$\sum_j (\sum_i c_{ij} e_i) \bigotimes f_j = 0$$

Hence

$$\sum_i c_{ij} e_i = 0$$

for all j, thus $c_{ij} = 0$ for i and j.

$\mathbb{R}^a$ has a basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 0 \\ 1 \\ . \\ . \\ 0 \end{pmatrix}$$

, ...

$\mathbb{R}^b$ has a basis

$$f_1 = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}$$

$$f_2 = \begin{pmatrix} 0 \\ 1 \\ . \\ . \\ 0 \end{pmatrix}$$

, ...

$\mathbb{R}^a \bigotimes \mathbb{R}^b$ has dimension ab with basis $e_i \bigotimes f_j$.

List the vectors in the basis as

$$e_1 \bigotimes f_1 = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}$$

,

$$e_1 \bigotimes f_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}$$

, ... ,

$$e_1 \bigotimes f_b = \begin{pmatrix} 0 \\ . \\ . \\ 0 \\ 1 \\ 0 \\ . \\ . \end{pmatrix}$$

, where 1 is in b's row,

$$e_2 \bigotimes f_1 = \begin{pmatrix} 0 \\ . \\ . \\ 0 \\ 1 \\ 0 \\ . \\ . \end{pmatrix}$$

, where 1 is in (b+1)'s row, ...

$$e_i \bigotimes f_j = \begin{pmatrix} \\ \\ - \\ f_j \\ - \\ \\ \\ \end{pmatrix}$$

, where f_j is in i's block.

In general,

$$\begin{pmatrix} x_1 \\ . \\ . \\ x_a \end{pmatrix} \bigotimes \begin{pmatrix} y_1 \\ . \\ . \\ y_b \end{pmatrix} = \begin{pmatrix} x_1 \begin{pmatrix} y_1 \\ . \\ . \\ y_b \end{pmatrix} \\ x_2 \begin{pmatrix} y_1 \\ . \\ . \\ y_b \end{pmatrix} \\ . \\ . \end{pmatrix}$$