4.5. Tensor Product (conti.)

We assume$R = \mathbb{R}$
Fact:If $e_1, e_2, \ldots, e_s$ are linear independent in the $\mathbb{R}$-module M and $n_1, n_2, \ldots, n_s \in N$ s.t. $e_1 \otimes n_1 + e_2 \otimes n_2 + \cdots + e_s \otimes n_s = 0$, then $n_1 = n_2 = \cdots = n_s = 0$(Theorem 5.11)
(If write element in $M \times N$ as a sum of $e_i \otimes n_i$,there is no way to reduce the length of the sum.)

Theorem. *$\{e_i\}_{i=1}^a$ is a basis of M and $\{f_i\}_{i=1}^b$ is a basis of N*
$\Rightarrow \{e_i \otimes f_j\}$ is a basis of $M \otimes N$

Proof. *Span:*

Pick any $m \times n \in M \otimes_R N$.
Suppose

$$m = \sum_{i=1}^{a} c_i e_i \text{ and } n = \sum_{j=1}^{b} d_j f_j$$

Then

$$M \otimes N = (\sum c_i e_i) \otimes (\sum d_j f_j) = \sum_i \sum_j c_i d_j e_i \otimes f_j$$

Linear independent:

Suppose

$$\sum_{i,j} c_{ij} e_i \otimes f_j = 0$$

Then

$$\sum_j (\sum_i C_{ij} e_i) \otimes f_j = 0$$

Hence

$$\sum_i c_{ij} e_i = 0 \ for \ all \ j$$

Thus $c_{ij} = 0$ for all j
for all i $\square$

$\mathbb{R}^a$ *has a basis* $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots$

$\mathbb{R}^b$ *has a basis* $f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots$

$\mathbb{R}_a \otimes \mathbb{R}_b$ *has dimension ab with basis* $e_i \otimes f_j$.
List the vertices in the basis as

$$\begin{array}{ccccccc} e_1 \otimes f_1, & e_2 \otimes f_2, & \ldots, & e_1 \otimes f_b, & e_2 \otimes f_1, & \ldots, & e_i \otimes f_j \\ \| & \| & & \| & \| & & \| \\ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix} & & b \rightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} & b+1 \rightarrow \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & & i\ block \rightarrow \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \frown \\ f_j \\ \smile \\ \vdots \\ 0 \end{pmatrix} \end{array}$$

$$Ingeneral, \begin{pmatrix} x_1 \\ \vdots \\ x_a \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ \vdots \\ y_b \end{pmatrix} = \begin{pmatrix} x_1 \begin{pmatrix} y_1 \\ \vdots \\ y_b \end{pmatrix} \\ x_2 \begin{pmatrix} y_1 \\ \vdots \\ y_b \end{pmatrix} \\ \vdots \end{pmatrix}$$