

## 4.6 Modules Over $PID$

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We assume  $D$  is a  $PID$  in this section.

**Theorem 1** *Let  $M$  be a free  $D$ -module of rank  $n$  and  $N$  is a  $D$ -submodule of  $M$ . Then  $N$  is a free module with  $\text{rank}(N) = m \leq n$ .*

PROOF. By induction on  $n$ . If  $n=0$ , then  $M = \{0\} = N$ . Hence  $\text{rank}(N) = 0 \leq n$ . In general, let  $\{e_1, e_2, e_3, \dots, e_n\}$  be a basis of  $M$ . If  $N \subseteq \text{span}_D\{e_2, e_3, \dots, e_n\}$ , then we have done by induction. So we suppose  $N \not\subseteq \text{span}_D\{e_2, e_3, \dots, e_n\}$ . Let  $I = \{a \in D \mid ae_1 + b \in N \text{ for some } b \in \text{span}_D\{e_2, e_3, \dots, e_n\}\}$ . Note  $I$  is an ideal of  $D$ . Then  $I = (d)$  for some  $d \in D$ , but  $I \neq (0)$ . Thus  $f_1 = de_1 + y_1 \in N$  for some  $y_1 \in \text{span}_D\{e_2, e_3, \dots, e_n\}$ . Let  $L = N \cap \text{span}_D\{e_2, e_3, \dots, e_n\}$  be a  $D$ -submodule of  $\text{span}_D\{e_2, e_3, \dots, e_n\}$ . By induction,  $L$  has a basis  $f_2, f_3, \dots, f_m$  where  $m \leq n$ . Claim  $f_1, f_2, f_3, \dots, f_m$  are linear independent. Suppose  $c_1f_1 + c_2f_2 + \dots + c_mf_m = 0$ . Notice that  $c_1f_1 = c_1de_1 + c_1y_1$  and  $c_1y_1 + c_2f_2 + \dots + c_mf_m \in \text{span}_D\{e_2, e_3, \dots, e_n\}$ . Hence  $c_1d = 0$ . Since  $D$  is an integral domain and  $d \neq 0$ . Then  $c_1 = 0$  and  $c_2 = c_3 = \dots = c_m = 0$ . Now claim  $f_1, f_2, f_3, \dots, f_m$  span  $N$ . Pick any  $n \in N$ . Then  $n = be_1 + y$  for  $b \in I$ ,  $y \in \text{span}_D\{e_2, e_3, \dots, e_n\}$ . Hence  $b = cd$  for some  $c \in D$ . Thus  $n - cf_1 = be_1 + y - c(de_1 + y_1) = cde_1 + y - cde_1 - cy_1 = y - cy_1 \in \text{span}_D\{e_2, e_3, \dots, e_n\} \cap N = L$ . Hence  $n - cf_1 = c_2f_2 + c_3f_3 + \dots + c_mf_m$  for some  $c_i \in D$ . Thus  $\{f_1, f_2, \dots, f_m\}$  is a basis of  $N$ .