

Fix an  $n \times n$  matrix A over  $\mathbb{R}$ .

We want to find all  $n \times n$  matrix B such that  $AB=BA$ .

We consider the  $\mathbb{R}[\lambda] - module \mathbb{R}^n$  determined by A.

Then  $\mathbb{R}^n = \mathbb{R}[\lambda]z_1 \oplus \cdots \oplus \mathbb{R}[\lambda]z_s$  where  $\mathbb{R}[\lambda] \neq O_{Z_1} \supseteq \cdots \supseteq O_{Z_s}$ .

**Theorem.** Let B be an  $n \times n$  matrix over  $\mathbb{R}$ . Define function  $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $B(u) = Bu$  for  $u \in \mathbb{R}^n$ . Then  $B \in Hom_{\mathbb{R}[\lambda]}(\mathbb{R}^n, \mathbb{R}^n) \Leftrightarrow AB = BA$ .

**Proof.** • ( $\Rightarrow$ )  $ABu = \lambda Bu = B\lambda u = BAu$  for any  $u \in \mathbb{R}^n$   
Then  $AB + BA$ .

• ( $\Leftarrow$ )

$$\begin{aligned} B(u + f(\lambda)v) &= B(u + f(A)v) \\ &= Bu + Bf(A)v \\ &= Bu + f(A)Bv \\ &= Bu + f(\lambda)Bv \end{aligned}$$

Hence  $B \in Hom_{\mathbb{R}[\lambda]}(\mathbb{R}^n, \mathbb{R}^n)$ .  $\square$

Suppose  $AB = BA$ . Then  $B \in Hom_{\mathbb{R}[\lambda]}(\mathbb{R}^n, \mathbb{R}^n)$ .

Suppose  $Bz_i = \sum_{j=1}^s C_{ij}z_j$  for some  $c_{ji} \in \mathbb{R}[\lambda]$ .

**Note.** 1.  $c_{ji}$  is not unique ; in fact if  $c'_{ji} = c_{ji} \text{ mod } d_j$ , then we can replace  $c_{ji}$  by  $c'_{ji}$ . Hence we can assume  $\deg(c_{ji}) < \deg(d_j)$

$$\begin{aligned} 2. \quad 0 &= Bd_i z_i = d_i Bz_i = \sum_{j=1}^s d_i c_{ji} z_j \\ &\Rightarrow d_i c_{ji} z_j = 0 \text{ for all } j \\ &\Rightarrow d_i c_{ji} \in O_{Z_j}(d_j) \\ &\Rightarrow d_j | d_i c_{ji}. \end{aligned}$$

**Example.**

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

Find B s.t.  $AB = BA$

**sol.**  $\mathbb{R}^3 = \mathbb{R}[\lambda]z_1 \oplus \mathbb{R}[\lambda]z_2$

$$d_1(\lambda) = \lambda - 1$$

$$d_2(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

$$z_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, z_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, z_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$C = \begin{pmatrix} c_{11} & c_{12} \\ (\lambda - 1)c_{21} & c_{22} + c'_{22}\lambda \end{pmatrix} \text{ for } c_{ij}, c'_{22} \in \mathbb{R}$$

$$\begin{aligned}
B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= Bz_1 = c_{11}z_1 + (\lambda - 1)c_{21}z_2 \\
&= \begin{pmatrix} c_{11} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -c_{21} \\ c_{21} \end{pmatrix} = \begin{pmatrix} c_{11} \\ -c_{21} \\ c_{21} \end{pmatrix} \\
B \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &= Bz_2 = c_{12}z_1 + (c_{22} + c'_{22}\lambda)z_2 \\
&= \begin{pmatrix} c_{12} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_{22} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c'_{22} \end{pmatrix} = \begin{pmatrix} c_{12} \\ c_{22} \\ c'_{22} \end{pmatrix} \\
B \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} &= B\lambda z_2 = \lambda Bz_2 \\
&= A \begin{pmatrix} c_{12} \\ c_{22} \\ c'_{22} \end{pmatrix} \begin{pmatrix} c_{12} \\ -c_{22} \\ c_{22} + 2c'_{22} \end{pmatrix} \\
Hence \quad B &= \begin{pmatrix} c_{11} & c_{12} & c_{12} \\ -c_{21} & c_{22} & -c'_{22} \\ c_{21} & c_{22} & c_{22} + 2c'_{22} \end{pmatrix}
\end{aligned}$$

**Note.**  $\{B|AB = BA\}$  has dimension 5  
and  $\{f(A)|f \in \mathbb{R}[\lambda]\}$  has dimension 2