Comments about the proof of SNF theorem

1. D is ED $\Rightarrow$ there exists $\delta : D - \{0\} \rightarrow \mathbb{N}$ s.t. for any $a, 0 \neq b$, in D, there exist $x \in$ D with $a = bx + r$, where $r = 0$ or $\delta(r) < \delta(b)$.
2. If we assume D is ED, we can use $l = \delta$, in the proof and find that the extra type is not necessary.

Theorem. Let M be a finitely generated D-module. Then $M \cong D\diagup_{(d_1)} \oplus \cdots \oplus D\diagup_{(d_r)}$, for some $d_i \in D$ with $d_1|d_2|\cdots|d_r$.

Proof. Let $x_1, x_2, \cdots, x_n$ be generators of M. Define a homomorphism $f : D^n \rightarrow M$ by $f\big((c_1, c_2, \cdots, c_n)\big) \rightarrow \sum_{i=1}^{n} c_i x_i$. Note f is is onto. Hence $M \cong D^n\diagup_{ker(f)}$. Let $\{f_1, f_2, \cdots, f_m\}$ be a basis of $ker(f)$ where $m \leq n$. Then

$$\begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} = A \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

for some $m \times n$ matrix A over D and $e_i = (0, 0, \cdots, 0, \overset{\text{ith position}}{1}, 0, \cdots)$
By SNF Theorem. there exist invertible $m \times m$ and $n \times n$ matries P and Q s.t. $PAQ = diag(1, 1, \cdots, 1, d_1 \cdots d_r)$. Set

$$\begin{bmatrix} f_1' \\ \vdots \\ f_m' \end{bmatrix} = P \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e_1' \\ \vdots \\ e_n' \end{bmatrix} = Q^{-1} \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

Then

$$\begin{bmatrix} f_1' \\ \vdots \\ f_m' \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & d_1 & & \\ & & & & \ddots & \\ & & & & & d_r \end{bmatrix}_{m \times n} \begin{bmatrix} e_1' \\ \vdots \\ e_n' \end{bmatrix}$$

Thus, $M \cong D\diagup_{(d_1)} \oplus \cdots \oplus D\diagup_{(d_r)}$ (factor decomposition)
Note that if $d_i = 0$ then $D\diagup_{(d_i)} \cong D$. $\square$

Note:

1. For $p, q \in \mathbb{N}$, with $gcd(p,q) = 1$.
2. For $p(\lambda), q(\lambda) \in \mathbb{R}[\lambda]$, the $\mathbb{R}[\lambda]$-module $\mathbb{R}[\lambda]/_{(p(\lambda))} \oplus \mathbb{R}[\lambda]/_{(q(\lambda))} \cong \mathbb{R}[\lambda]/_{(p(\lambda)q(\lambda))}$, if $gcd(p(\lambda), q(\lambda)) = 1$.

Example.

$$
\begin{aligned}
M &= \mathbb{Z}_2 \oplus \mathbb{Z}_{72} \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_{81} \oplus \mathbb{Z}_{125} \\
\Rightarrow M &\cong \mathbb{Z}_2 \oplus \big(\mathbb{Z}_8 \oplus \mathbb{Z}_9\big) \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_{81} \oplus \mathbb{Z}_{125} \\
\Rightarrow M &\cong \underbrace{\big(\mathbb{Z}_2 \oplus \mathbb{Z}_8\big)}_{M_1} \oplus \underbrace{\big(\mathbb{Z}_9 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_{81}\big)}_{M_2} \oplus \underbrace{\mathbb{Z}_{125}}_{M_3} \quad \textbf{\textit{(primary decomposition)}} \\
& \quad 2, 8, 9, 27, 81, 125 \text{ are called elementary divisors of M} \\
\Rightarrow M &\cong \big(\mathbb{Z}_9\big) \oplus \big(\mathbb{Z}_2 \oplus \mathbb{Z}_{27}\big) \oplus \big(\mathbb{Z}_8 \oplus \mathbb{Z}_{81} \oplus \mathbb{Z}_{125}\big) \\
\Rightarrow M &\cong \mathbb{Z}_9 \oplus \mathbb{Z}_{54} \oplus \mathbb{Z}_{81000} \quad \textbf{\textit{(factor decomposition)}} \\
& \quad 9, 54, 81000 \text{ are called invariant factors of M}
\end{aligned}
$$

Fact: Elementary divisors and invariant factors of M are unique.

Example.

$$
\begin{aligned}
M &= \mathbb{R}[\lambda]/_{((\lambda-1)^3)} \oplus \mathbb{R}[\lambda]/_{((\lambda^2+1)^2)} \oplus \mathbb{R}[\lambda]/_{((\lambda-1)(\lambda^2+1)^4)} \oplus \mathbb{R}[\lambda]/_{((\lambda+2)(\lambda^2+1)^2)} \quad \text{over} \quad \mathbb{R}[\lambda] \\
&\cong \big(\mathbb{R}[\lambda]/_{((\lambda-1))} \oplus \mathbb{R}[\lambda]/_{((\lambda-1)^3)}\big) \oplus \big(\mathbb{R}[\lambda]/_{((\lambda+2))}\big) \oplus \\
&\quad \big(\mathbb{R}[\lambda]/_{((\lambda^2+1)^2)} \oplus \mathbb{R}[\lambda]/_{((\lambda^2+1)^2)} \oplus \mathbb{R}[\lambda]/_{((\lambda^2+1)^4)}\big) \quad \text{(primary decomposition)} \\
&\quad (\lambda-1), (\lambda-1)^3, (\lambda+2), (\lambda^2+1)^2, (\lambda^2+1)^2, (\lambda^2+1)^4 \text{ are called elementary divisors of M} \\
&\cong \big(\mathbb{R}[\lambda]/_{((\lambda^2+1)^2)}\big) \oplus \big(\mathbb{R}[\lambda]/_{((\lambda^2+1)^2)} \oplus \mathbb{R}[\lambda]/_{((\lambda-1))}\big) \oplus \\
&\quad \big(\mathbb{R}[\lambda]/_{((\lambda^2+1)^4)} \oplus \mathbb{R}[\lambda]/_{((\lambda-1)^3)} \oplus \mathbb{R}[\lambda]/_{((\lambda+2))}\big) \\
&\cong \big(\mathbb{R}[\lambda]/_{((\lambda^2+1)^2)}\big) \oplus \big(\mathbb{R}[\lambda]/_{((\lambda^2+1)^2(\lambda-1))}\big) \oplus \big(\mathbb{R}[\lambda]/_{((\lambda^2+1)^4(\lambda-1)^3(\lambda+2))}\big) \quad \text{(factor decomposition)} \\
&\quad (\lambda^2+1)^2, (\lambda^2+1)^2(\lambda-1), (\lambda^2+1)^4(\lambda-1)^3(\lambda+2) \text{ are called invariant factors of M}
\end{aligned}
$$