Advanced Algebra I Class Note

4.6 Modules over a Principal Ideal Domain–Structure Theorey

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Rational Canonical form of a matrix.

Fix an $n \times n$ matrix A over $\mathbb{R}$.

Let $\mathbb{R}^n$ be a $\mathbb{R}[\lambda]$-module defined by $\lambda u = Au$ for $u \in \mathbb{R}^n$.

We want to study the structure of $\mathbb{R}^n$.

Note

$e_i = (0, ..., 0, 1, 0, ..., 0)^t \in \mathbb{R}^n$, where 1 is in the ith position and $1 \le i \le n$, is a basis over $\mathbb{R}$, but not over $\mathbb{R}[\lambda]$.

Lemma

$\mathbb{R}^n$ is a torsion $\mathbb{R}[\lambda]$-module.

Proof

Choose any $u \in \mathbb{R}^n$.

Then u, λu, $\lambda^2 u$, ..., $\lambda^n u$ are not linearly independent over $\mathbb{R}$.

Then $f(\lambda) u = 0$ for some $0 \ne f(\lambda) \in \mathbb{R}[\lambda]$.

Set $u_i = (0, ..., 0, 1, 0, ..., 0)^t \in (\mathbb{R}[\lambda])^n$, where 1 is in the ith position.

Then $\{u_1, ..., u_n\}$ is a basis of $(\mathbb{R}[\lambda])^n$.

Define a map $\eta : (\mathbb{R}[\lambda])^n \longrightarrow \mathbb{R}^n$ by $\eta\left(\sum\limits_{i=1}^{n} c_i u_i\right) = \sum\limits_{i=1}^{n} c_i e_i$.

Then η is a surjective homomorphism.

Hence $\mathbb{R}^n \cong \dfrac{(\mathbb{R}[\lambda])^n}{\mathtt{ker}\eta}$ as $\mathbb{R}[\lambda]$-module.

Set $\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} := (\lambda I - A^t) \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$.

Lemma

$\{f_1, f_2, ..., f_n\}$ is a basis of $\texttt{ker}\eta$.

Proof

Note $\texttt{ker}\eta$ is a free $\mathbb{R}[\lambda]$-submodule of $(\mathbb{R}[\lambda])^n$ with rank$\leq n$.

It suffices to show $f_1, ..., f_n$ are in $\texttt{ker}\eta$ and are linearly independent.

$$\begin{aligned}\text{Observe } \eta(f_i) &= \eta\left(\lambda u_i - \sum_{j=1}^{n} a_{ji}u_j\right)\\ &= \lambda e_i - \sum_{j=1}^{n} a_{ji}e_j\\ &= Ae_i - \sum_{j=1}^{n} a_{ji}e_j\\ &= (i\text{th colume of } A) - (a_{1i}e_1 + a_{2i}e_2 + \cdots + a_{ni}e_n)\\ &= 0.\end{aligned}$$

Hence $f_i \in \texttt{ker}\eta$.

Suppose $\sum_{j=1}^{n} h_i f_i = 0$ for $h_i \in \mathbb{R}[\lambda]$.

$$\begin{aligned}\text{Then } 0 &= \sum_{j=1}^{n} h_i\left(\lambda u_i - \sum_{j=1}^{n} a_{ji}u_j\right)\\ &= \sum_{j=1}^{n} h_i\lambda u_i - \sum_{i=1}^{n}\sum_{j=1}^{n} h_i a_{ji}u_j\\ &= \sum_{j=1}^{n} h_i\lambda u_i - \sum_{i=1}^{n}\sum_{j=1}^{n} h_j a_{ij}u_i\\ &= \sum_{j=1}^{n}\left(h_i\lambda - \sum_{j=1}^{n} h_j a_{ij}\right)u_i.\end{aligned}$$

Since $\{u_i\}$ is a basis, $\lambda h_i - \sum_{j=1}^{n} a_{ij}h_j = 0$ for all i.

Suppose h_r has maximun degree among h_i.

$$\begin{aligned}\text{Then } \deg(\lambda h_r) + 1 = \deg(\lambda h_r) &= \deg\left(\sum_{j=1}^{n} a_{ij}h_j\right)\\ &\leq \deg(h_r), \text{ a contradiction.}\end{aligned}$$

Then $h_i = 0$ for all i.

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Recall

Fix an $n \times n$ matrix A over $\mathbb{R}$.

Consider $\mathbb{R}^n$ as a $\mathbb{R}[\lambda]$-module determined by A.

We want to study the structure of $\mathbb{R}^n$ as a torsion $\mathbb{R}[\lambda]$-module.

Define a map $\eta : (\mathbb{R}[\lambda])^n \longrightarrow \mathbb{R}^n$ by $\eta\left(\sum_{i=1}^{n} c_i u_i\right) = \sum_{i=1}^{n} c_i e_i$,

where $\{u_i\}$ and $\{e_i\}$ are standard bases of $(\mathbb{R}[\lambda])^n$ and $\mathbb{R}^n$ respectively.

Set $\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} := (\lambda I - A^t) \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$.

Then $\mathtt{ker}\eta$ has a basis $\{f_1, f_2, ..., f_n\}$ over $\mathbb{R}[\lambda]$.

Choose invertable matrices P, Q such that $P(\lambda I - A^t) Q = \begin{bmatrix} 1 & 0 & & & & 0 \\ 0 & \ddots & \ddots & & & \\ & \ddots & 1 & 0 & & \\ & & 0 & d_1 & \ddots & \\ & & & \ddots & \ddots & 0 \\ 0 & & & & 0 & d_s \end{bmatrix}$

such that $d_1 \mid d_2 \mid \cdots \mid d_s$.

Then $P \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = P(\lambda I - A^t) Q Q^{-1} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$.

Set $\begin{bmatrix} f_1' \\ \vdots \\ f_n' \end{bmatrix} = P \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$, $\begin{bmatrix} v_1 \\ \vdots \\ v_{n-s} \\ z_1 \\ \vdots \\ z_s \end{bmatrix} = Q^{-1} \begin{bmatrix} u_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ u_n \end{bmatrix}$.

Then $f_i' = v_i$ for $1 \le i \le n-s$ and $f_{s+i}' = d_i z_i$ for $1 \le i \le s$.

Hence $\dfrac{(\mathbb{R}[\lambda])^n}{\mathtt{ker}\eta} \cong \mathbb{R}^n$

$= \mathbb{R}[\lambda]\,\eta(z_1) \oplus \cdots \oplus \mathbb{R}[\lambda]\,\eta(z_s)$, and $O_{z_i} = (d_i)$.

Fix $1 \le i \le s$, observe $\eta(z_i)$, $\lambda\eta(z_i)$, ..., $\lambda^{deg(d_i)-1}\eta(z_i)$ is a basis over $\mathbb{R}$,

and $A\lambda^j\eta(z_i) = \lambda^{j+1}\eta(z_i)$ if $j < deg(d_i) - 1$.

Note $A\lambda^{deg(d_i)-1}\eta(z_i) = \lambda^{deg(d_i)}\eta(z_i) = \left(\lambda^{deg(d_i)} - d_i(\lambda)\right)\eta(z_i)$,

with $d_i(\lambda) = \eta(z_i) = 0$,

where $d_i(\lambda) = \lambda^{n_i} - c_{i(n_i-1)}\lambda^{n_i-1} + \cdots + c_{i1}\lambda + c_{i0}.(n_i = deg(d_i))$

$AS = S\Lambda$.

Then $S^{-1}AS = \Lambda$,

$$\text{where } S = \begin{bmatrix} \\ \cdots & \eta(z_i) & \lambda\eta(z_i) & \cdots & \lambda^{n_i-1}\eta(z_i) & \cdots \\ \\ \end{bmatrix}$$

$$\text{and } \Lambda = \begin{bmatrix}
\square & & & & & & & & & \\
 & & & & 0 & & & & 0 & \\
 & \square & & & & & & & & \\
 & & 0 & 0 & & & & -c_{i0} & & \\
 & & 1 & 0 & & & & -c_{i1} & & \\
 & & & 1 & 0 & & & \vdots & & \\
0 & & & & \ddots & \ddots & & \vdots & 0 & \\
 & & & & & 1 & 0 & \vdots & & \\
 & & 0 & & & & 1 & -c_{i(n_i-1)} & & \\
 & & & & & & & & \square & \\
0 & & & & 0 & & & & & \square
\end{bmatrix}.$$

Definition

A matrix of the form Λ is called the rational canonical form of A.

Definition

Two $n \times n$ matrices A, B are similar if $B = S^{-1}AS$ for some invertable matrix S.

Example

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 2 & -1 \\ 6 & 3 & 4 \end{bmatrix}.$$

$$\lambda I - A^t = \begin{bmatrix} \lambda+1 & 2 & -6 \\ 1 & \lambda & -3 \\ 1 & 1 & \lambda-4 \end{bmatrix}$$

$$\xrightarrow{P_1=\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} \lambda-1 & 2-2\lambda & 0 \\ 0 & \lambda-1 & 1-\lambda \\ 1 & 1 & \lambda-4 \end{bmatrix}$$

$$\xrightarrow{P_2=\begin{bmatrix} 1 & -2 & 1-\lambda \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 0 & 3-3\lambda & (\lambda-4)(1-\lambda) \\ 0 & \lambda-1 & 1-\lambda \\ 1 & 1 & \lambda-4 \end{bmatrix}$$

$$\xrightarrow{Q_1=\begin{bmatrix} 1 & -1 & 4-\lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 0 & 3-3\lambda & (\lambda-4)(1-\lambda) \\ 0 & \lambda-1 & 1-\lambda \\ 1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{Q=Q_2=\begin{bmatrix} 1 & -1 & 3-\lambda \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 0 & 3-3\lambda & -(\lambda-1)^2 \\ 0 & \lambda-1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{P_3=\begin{bmatrix}1 & 1 & -2-\lambda\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}} \begin{bmatrix}0 & 0 & -(\lambda-1)^2\\ 0 & \lambda-1 & 0\\ 1 & 0 & 0\end{bmatrix}$$

$$\xrightarrow{P=P_4=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & -1\\ -1 & -1 & 2+\lambda\end{bmatrix}} \begin{bmatrix}1 & 0 & 0\\ 0 & \lambda-1 & 0\\ 0 & 0 & (\lambda-1)^2\end{bmatrix}.$$

Then $P\left(\lambda I - A^t\right)Q = diag\left(1, \lambda-1, (\lambda-1)^2\right)$.

Hence $\Lambda = \begin{bmatrix}1 & 0\\ 0 & \begin{matrix}0 & -1\\ 1 & 2\end{matrix}\end{bmatrix}.$

Find Q^{-1}:

$$\begin{bmatrix}I & Q\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1 & -1 & 3-\lambda\\ 0 & 1 & 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 0 & 0 & 1\end{bmatrix}$$

$$\longrightarrow \cdots \longrightarrow \begin{bmatrix}1 & 1 & \lambda-4 & 1 & 0 & 0\\ 0 & 1 & -1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}Q^{-1} & I\end{bmatrix}.$$

$$\begin{bmatrix}v_1\\ z_1\\ z_2\end{bmatrix} = Q^{-1}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} = \begin{bmatrix}u_1+u_2+(\lambda-4)u_3\\ u_2-u_3\\ u_3\end{bmatrix} \in \left(\mathbb{R}[\lambda]\right)^3.$$

$$\eta(z_1) = \eta(u_2) - \eta(u_3) = e_2 - e_3 = \begin{bmatrix}0\\ 1\\ -1\end{bmatrix}.$$

$$\eta(z_2) = \eta(u_3) = e_3 = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}.$$

$$\lambda\eta(z_2) = A\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\\4\end{bmatrix}.$$

Hence $S = \begin{bmatrix}0 & 0 & -1\\1 & 0 & -1\\-1 & 1 & 4\end{bmatrix}$ and $SAS^{-1} = \Lambda$.

Note

$$\eta(z_1) = \begin{bmatrix}0\\1\\-1\end{bmatrix},\ \eta(z_2) = \begin{bmatrix}0\\0\\1\end{bmatrix},\ (\lambda - 1)\eta(z_2) = \begin{bmatrix}-1\\-1\\3\end{bmatrix}$$

is another basis of $\mathbb{R}^3$ over $\mathbb{R}$

and $(A - I)\eta(z_1) = 0$,

$(A - I)\eta(z_2) = (A - I)\eta(z_2)$,

$(A - I)(A - I)\eta(z_2) = 0$.

Hence $A\eta(z_1) = \eta(z_1)$,

$A\eta(z_2) = \eta(z_2) + (A - I)\eta(z_2)$,

$A(A - I)\eta(z_2) = (A - I)\eta(z_2)$.

Thus $A\begin{bmatrix}0 & 0 & -1\\1 & 0 & -1\\-1 & 1 & 3\end{bmatrix} = \begin{bmatrix}0 & 0 & -1\\1 & 0 & -1\\-1 & 1 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 1\end{bmatrix}$,

where $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 1\end{bmatrix}$ is called the Jordan canonical form of A.

Note

If $\lambda I - A^t$ has simith normal form $diag\left(1, 1, ..., 1, (\lambda - 1)(\lambda - 2), (\lambda - 1)(\lambda - 2)^2(\lambda - 3)\right)$, then $\lambda - 1$, $\lambda - 1$, $\lambda - 2$, $(\lambda - 2)^2$, $\lambda - 3$ are elementary divisors and the Jordan canonical form is $\begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 2 & & & \\ & & & 2 & 0 & \\ & & & 1 & 2 & \\ 0 & & & & & 3 \end{bmatrix}$

with respect to the basis $(A-2I)\,\eta\,(z_1)$, $(A-2I)^2\,(A-3I)\,\eta\,(z_2)$,
$(A-I)\,\eta\,(z_3)$, $(A-I)\,(A-3I)\,\eta\,(z_2)$, $(A-I)\,(A-2I)^2\,\eta\,(z_3)$.

Note

d_s is the minimal polynomial of A.

$$\begin{bmatrix} 1 & 0 & & & 0 & \\ 0 & 2 & & & & \\ & & 1 & & & 0 \\ & & & 2 & 0 & \\ & 0 & & 1 & 2 & \\ & & 0 & & & 3 \end{bmatrix}.$$