進階代數(上) 第一次作業

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We always assume that a ring has the multiplication identity 1.

1. Let Q denote the ring of real quarternions. For $x = a + bi + cj + dk \in Q$ the *conjugate* of x is $x^* := a - bi - cj - dk$.
   (a) (連敏筠) Show $(a+bi+cj+dk)(a-bi-cj-dk) = a^2+b^2+c^2+d^2$ for $a, b, c, d \in \mathbb{R}$.
   (b) (施智懷) Suppose $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in \mathbb{Z}$. Show that there exist $a, b, c, d \in \mathbb{Z}$ such that
   $$(a_1^2 + b_1^2 + c_1^2 + d_1^2)(a_2^2 + b_2^2 + c_2^2 + d_2^2) = a^2 + b^2 + c^2 + d^2.$$
   (c) (邱鈺傑) Suppose $u \in \mathbb{Z}$ and $2u = a^2 + b^2 + c^2 + d^2$ for some $a, b, c, d \in \mathbb{Z}$. Then $u = e^2 + f^2 + g^2 + h^2$ for some $e, f, g, h \in \mathbb{Z}$. (Hint. Try $e = (a + b)/2$ and $f = (a - b)/2$.)

   設計理論會用到一個不太容易證的定理: 任意正整數都能寫成四整數的平分和. 你能猜測此定理證明的方向嗎?

2. Let R be a commutative ring of prime characteristic p.
   (a) (蕭雯華) Show that
   $$(a + b)^{p^n} = a^{p^n} + b^{p^n}$$
   for all $a, b \in \mathbb{N}$.
   (b) (婓若宇) Show that the map $f : R \to R$ given by $f(a) = a^p$ is a homomorphism of rings.

3. (林逸軒) An element a of a ring is *nilpotent* if $a^n = 0$ for some n. Prove that in a commutative ring $a + b$ is nilpotent if a and b are. Show that this result may be false if R is not commutative.

4. (陳巧玲) In a ring R show that the following conditions are equivalent.
   (a) R has no nonzero nilpotent elements.
   (b) If $a \in R$ and $a^2 = 0$, then $a = 0$.

5. (李光祥) Give an example of a nonzero homomorphism $f : R \to R'$ of rings such that $f(1) \neq 1'$. Is it possible $1'$ in the image of f?

6. (林詒琪) Find a nonidentity isomorphism ϕ of $\mathbb{R}$ into $\mathbb{R}$.

7. (葉彬) Show that the only ring homomorphism ϕ of $\mathbb{R}$ into $\mathbb{R}$ with $\phi(1) = 1$ is the identity.