進階代數(上) 第十次作業

上課老師: 翁志文

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1. (連敏筠) Let $a+b\sqrt{-1}$ be a nonzero element in the $\mathbb{Z}[\sqrt{-1}]$-module (not $\mathbb{Z}$-module) $\mathbb{Z}[\sqrt{-1}]$. Show that $|\mathbb{Z}[\sqrt{-1}]/(a+b\sqrt{-1})| = a^2+b^2$. (Hint Prove $|\mathbb{Z}\times\mathbb{Z}/((a,b),(-b,a))| = a^2+b^2$ first.)

2. Let A be an $n\times n$ matrix over $\mathbb{R}$. $\phi(\lambda) = \det(\lambda I - A)$ is called the *characteristic polynomial* of A. The *minimal polynomial* $m(\lambda)\in\mathbb{R}[\lambda]$ of A is the minic polynomial of least degree such that $m(A)=0$. Suppose $\lambda I - A^t$ has Smith normal form $\mathrm{diag}(d_1(\lambda), d_2(\lambda), \ldots, d_n(\lambda))$.
   (a) (施智懷) Determine the characteristic polynomial $\phi(\lambda)$ of A.
   (b) (邱鈺傑) Determine the minimal polynomial $m(\lambda)$ of A.
   (c) (裴若宇) Show that $\phi(A)=0$.

3. (蕭雯華) Show that if a square matrix A satisfying $A^2=A$ then A is similar to
$$\mathrm{diag}\{1,\ldots,1,0,\ldots,0\}.$$

4. (林逸軒) Determine the 4×4 matrices B over $\mathbb{R}$ commuting with
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

5. (陳巧玲) Determine the 5×5 matrices B over $\mathbb{R}$ commuting with
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

6. Let A be an $n\times n$ matrix over $\mathbb{R}$. Consider the following two statements:

(i) The matrices commuting with A have the form $f(A)$ for $f(\lambda)\in\mathbb{R}[\lambda]$;

(ii) The minimum polynomial of A is the characteristic polynomial of A.
   (a) (李光祥) Show (i)$\Rightarrow$ (ii).
   (b) (林詒琪) Show (ii)$\Rightarrow$ (i).