Solution for Homework 10

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1. We define $f : \mathbb{Z} \times \mathbb{Z}/((a, b), (-b, a)) \mapsto \mathbb{Z}[\sqrt{-1}]/(a + bi)$ by
$f([x, y]) = [x + yi]$. We claim that f is bijection. Then we check
Onto: Clear.
One to one: Suppose $\exists [(x, y)], [(x', y')] \in \mathbb{Z} \times \mathbb{Z}/((a, b), (-b, a))$ such that
$f([(x, y)]) = x + yi = f([(x', y')]) = x' + y'i$. Then we have
$(x - x') + (y - y') = 0$.
Thus there exists some s, t, a, b such that $(s + ti)(a + bi) = 0$ and
$x - x' = sa - tb$, $y - y' = sb + ta$.
$\Rightarrow (x - x', y - y') = s(a, b) + t(-b, a) \in ((a, b), (-b, a))$
$\Rightarrow [x - x', y - y'] = 0$. Therefore $x = x'$, $y = y'$.
Well-define is similar to prove.
Now since $(a, b)\dot{(}-b, a) = 0$, $(a, b) \perp (-b, a)$
Thus $\| (a, b) \| \times \| (-b, a) \| = a^2 + b^2$. $\blacksquare$

Comment from teacher:
$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ has smith normal form $\begin{pmatrix} 1 & 0 \\ 0 & a^2 + b^2 \end{pmatrix}$. Hence
$\mathbb{Z} \times \mathbb{Z}/((a, b), (-b, a)) \cong \mathbb{Z}_{a^2+b^2}$ and $| \mathbb{Z}_{a^2+b^2} | = a^2 + b^2$.

2.(a) $\phi(\lambda) = det(\lambda I - A) = det((\lambda I - A)^t) = det(\lambda I - A^t) = det(PDQ)$ for some matrix P, D, Q where D is a diagonal matrix, P, Q are unit. Thus $det(D) = d_1(\lambda)...d_n(\lambda)$.

(b) Since $A \sim B$ (A is similar to B) we have $min(A) = min(B)$.

$$A \sim \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_n \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} B_1^2 & & 0 \\ & \ddots & \\ 0 & & B_n^2 \end{pmatrix}$$

(Since $A = S^{-1}BS \Rightarrow A^2 = S^{-1}BSS^{-1}BS = S^{-1}B^2S$, so is A^3, A^4...)

$$\text{Thus } f(A) \sim \begin{pmatrix} f(B_1) & & 0 \\ & \ddots & \\ 0 & & f(B_n) \end{pmatrix}.$$

We claim that (i) $d_n(A) = 0$ and (ii) $deg(f) < deg(d_n) \Rightarrow f(A) \neq 0$. To prove (i), let $d_n = (x - \alpha_1)^{n_1}(x - \alpha_2)^{n_2} \cdots (x - \alpha_t)^{n_t}(x^2 + a_1x + b_1)^{m_1}(x^2 + a_2x + b_2)^{m_2} \cdots (x^2 + a_sx + b_s)^{m_s}$, then $d_n =$
$(x - \alpha_1)^{n_1}(x - \alpha_2)^{n_2} \cdots (x - \alpha_t)^{n_t}(x - \lambda_1)^{m_1}(x - \overline{\lambda_1})^{m_1} \cdots (x - \lambda_s)^{m_s}(x - \overline{\lambda_s})^{m_s}$,
transform B_n into Jordan Canonical form we have

$$B_n = \begin{pmatrix} \alpha_1 & & & & & & & & & \\ 1 & \ddots & & & & & & & & \\ 0 & \ddots & \alpha_1 & & & & & & & \\ & & & \lambda_1 & & & & & & \\ & & & 1 & \ddots & & & & & \\ & & & & \ddots & \lambda_1 & & & & \\ & & & & & & \overline{\lambda_1} & & & \\ & & & & & & 1 & \ddots & & \\ & & & & & & & \ddots & \overline{\lambda_1} & \\ & & & & & & & & & \ddots \end{pmatrix}$$

where the number of α_i, λ_i, $\overline{\lambda_i}$ is n_i, m_i, m_i respectively.
Thus the minimal polynomial is
$(x-\alpha_1)^{n_1}(x-\alpha_2)^{n_2}\cdots(x-\alpha_t)^{n_t}(x-\lambda_1)^{m_1}(x-\overline{\lambda_1})^{m_1}\cdots(x-\lambda_s)^{m_s}(x-\overline{\lambda_s})^{m_s}$,
$\Rightarrow d_n(B_n) = 0$ (Note $d_1 \mid d_2 \mid \cdots \mid d_n$)
$\Rightarrow d_i(B_i) = 0$
$\Rightarrow d_n(B_i) = 0 \ \forall i$
$\Rightarrow d_n(A) = 0$
To prove (ii), note that

$$B_n = \begin{pmatrix} * & & & & * \\ 1 & \ddots & & & \\ & \ddots & & & \\ & & & & \\ 0 & & & 1 & * \end{pmatrix}, \ B_n^2 = \begin{pmatrix} * & & & & * \\ * & \ddots & & & \\ 1 & \ddots & & & \\ & \ddots & & & \\ & & & & \\ 0 & & 1 & * & * \end{pmatrix},$$

$$B_n^3 = \begin{pmatrix} * & & & & & * \\ * & \ddots & & & & \\ * & \ddots & & & & \\ 1 & \ddots & & & & \\ & \ddots & & & & \\ 0 & & 1 & * & * & * \end{pmatrix}$$
where * means it could be any value. Observe the behavior of the all 1 oblique line of the bottom-left triangle of the matrices. If $f(B_n) = c_{k-1}B_n^{k-1} + c_{k-2}B_n^{k-2} + \cdots + c_1 B_n^1 + c_0 I = 0$, then consider the bottom-left triangle of $f(B_n)$ we would find that f must be the constant polynomial $f(x) = 0$, which has no degree, a contradiction.

(c) By part (a), $\phi(\lambda) = d_1(\lambda)\cdots d_n(\lambda)$, so $\phi(A) = d_1(A)\cdots d_n(A)$. By part (b) $d_n(A) = 0$, thus $\phi(A) = 0$. ∎

3. By finding the rational canonical form of A, $A = SBS^{-1}$,

$$B = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_t \end{pmatrix}, \text{ where } B_i = \begin{pmatrix} 0 & & & & c_{i_1} \\ 1 & \ddots & & & \vdots \\ & \ddots & & & \\ 0 & & 1 & 0 & c_{i_{n_i}} \end{pmatrix}_{n_i \times n_i}.$$

Since $A^2 = A$, then $B^2 = B$ and therefore $B_i^2 = B_i$. Suppose

(i) $n_i = 1$

Then $B_i = (c_{i_1})$. Since $B^2 = B$, c_{i_1} must 0 or 1.(Because it's over an PID)

(ii) $n_i \geq 2$

If B_i are all zero then it's trivial. If not, we have

$(B_i)_{j_{n_i-1}} = 0\ \forall j = 1, 2, ..., n_i - 1$ and $(B_i^2)_{j_{n_i-1}} = c_{i_j} = 0\ \forall j = 1, 2, ..., n_i - 1$.

Suppose

(1) $n_i = 2$

Then $B_i = \begin{pmatrix} 0 & c_{i_1} \\ 1 & c_{i_2} \end{pmatrix}$ and $B_i^2 = \begin{pmatrix} c_{i_1} & c_{i_1}c_{i_2} \\ c_{i_2} & c_{i_1} + c_{i_2}^2 \end{pmatrix}$

$\Rightarrow c_{i_1} = 0, c_{i_2} = 1$ thus $B_i = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$.

Note that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ are inverse to each other.

Thus $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. By reordering the columns, we have

$A = diag\{1, ...1, 0, ..., 0\}$.

(2) $n_i \geq 3$

This is impossible. $\blacksquare$

Comment from teacher: $A^2 = A$ implies that A has minimal polynomial $x^2 - x$ or x or $x - 1$. But $xI - A^t \sim diag\{d_1, ..., d_n\}$ and d_n is the minimal polynomial, thus $xI - A^t =$

$$\begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix} \text{ or } \begin{pmatrix} x-1 & & \\ & \ddots & \\ & & x-1 \end{pmatrix} \text{ or}$$

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & x^2 - x & & \\ & & & & \ddots & \\ & & & & & x^2 - x \end{pmatrix}$$

4. Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

Since A is in rational canonical form, the 1 at top-left implies $d_1(\lambda) = \lambda - 1$ and the 4th column implies $d_2(\lambda) = (\lambda - 1)^3$.

Thus $\eta(z_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\eta(z_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\lambda\eta(z_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\lambda^2\eta(z_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

$\Rightarrow s = 2$ and c_{ji} fits $d_j \mid d_i c_{ji}$ and $deg(c_{ji}) < deg(d_j)$

$\Rightarrow C = \begin{pmatrix} a & b \\ (\lambda-1)^2 c & d + e\lambda + f\lambda^2 \end{pmatrix}$ for some $a, b, c, d, e, f \in \mathbb{R}$

$\Rightarrow Bz_i = \sum_{i=1}^{s} c_{ji} z_j$. By some routine calculating we have

$B = \begin{pmatrix} a & b & b & b \\ c & d & f & e+3f \\ -2c & e & d-3f & -3e-8f \\ c & f & e+3f & d+3e+6f \end{pmatrix}$ ■

5. Let $A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = (e_2,\ 0,\ e_4,\ e_5,\ 0) = \begin{pmatrix} 0 \\ e_1^t \\ 0 \\ e_3^t \\ e_4^t \end{pmatrix}$.

Let $B = (B^{(1)},\ B^{(2)},\ B^{(3)},\ B^{(4)},\ B^{(5)}) = \begin{pmatrix} B_{(1)} \\ B_{(2)} \\ B_{(3)} \\ B_{(4)} \\ B_{(5)} \end{pmatrix}$, where $B^{(i)}$ denotes the ith columns of B, $B_{(j)}$ denotes the jth row of B.

Since $AB = BA$,

$BA = \begin{pmatrix} B_{(1)} \\ B_{(2)} \\ B_{(3)} \\ B_{(4)} \\ B_{(5)} \end{pmatrix} (e_2,\ 0,\ e_4,\ e_5,\ 0) = \begin{pmatrix} B_{12} & 0 & B_{14} & B_{15} & 0 \\ B_{22} & 0 & B_{24} & B_{25} & 0 \\ B_{32} & 0 & B_{34} & B_{35} & 0 \\ B_{42} & 0 & B_{44} & B_{45} & 0 \\ B_{52} & 0 & B_{54} & B_{55} & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 0 \\ e_1^t \\ 0 \\ e_3^t \\ e_4^t \end{pmatrix} (B^{(1)},\ B^{(2)},\ B^{(3)},\ B^{(4)},\ B^{(5)}) =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\ 0 & 0 & 0 & 0 & 0 \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} \\ B_{41} & B_{42} & B_{43} & B_{44} & B_{45} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\ 0 & 0 & 0 & 0 & 0 \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} \\ B_{41} & B_{42} & B_{43} & B_{44} & B_{45} \end{pmatrix} = \begin{pmatrix} B_{12} & 0 & B_{14} & B_{15} & 0 \\ B_{22} & 0 & B_{24} & B_{25} & 0 \\ B_{32} & 0 & B_{34} & B_{35} & 0 \\ B_{42} & 0 & B_{44} & B_{45} & 0 \\ B_{52} & 0 & B_{54} & B_{55} & 0 \end{pmatrix}$$

$$\Rightarrow B = \begin{pmatrix} B_{11} & 0 & B_{13} & 0 & 0 \\ B_{21} & B_{11} & B_{23} & B_{13} & 0 \\ 0 & 0 & B_{33} & 0 & 0 \\ B_{41} & 0 & 0 & B_{33} & 0 \\ B_{51} & B_{41} & B_{53} & B_{54} & B_{33} \end{pmatrix} \text{ with dimension 9. } \blacksquare$$

6. (a) Let $X = \{B \in \mathbb{M}_{n\times n}(\mathbb{R}) \mid AB = BA\}$, $Y = \{B \in \mathbb{M}_{n\times n}(\mathbb{R}) \mid B = f(A)$ for some $f(\lambda) \in \mathbb{R}[\lambda]\}$

Claim: $dim(X) \geq n$ and $dim(Y) \leq n$

(1) Consider $\lambda I - A^t \sim \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & d_1 & & \\ & & & & \ddots & \\ & & & & & d_s \end{pmatrix}$, where

$\sum_{i=1}^{s} deg(d_i) = n$, then C, the matrix which is commuting to A is like

$C = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_s \end{pmatrix}_{s\times s}$ where α_i are blocks. The ith block contains $deg(d_i)$ variables, thus dimension of X is at least n.

(2) By the definition of minimal polynomial $m(\lambda)$ of A,
$dim(Y) = deg(m(\lambda)) \leq n$.

Then by the hypothesis we have $dim(X) = dim(Y)$, i.e. $deg(m(\lambda)) = n$, this implies that $m(\lambda) = \phi(\lambda)$.

(b) We define X and Y as in part (a). Then clearly $Y \subseteq X$ and by hypothesis we have $dim(Y) = n$.

Let $c = (c_i)$, $\{z_1, Az_1, ..., A^{n-1}z_1\}$ be a basis of Y. Let
$Bz_1 = c_0B_0 + c_1Az_1 + ... + c_{n-1}A^{n-1}z_1$. $(B \in X)$

Then $B(Az_1) = A(Bz_1) = c_0Az_1 + c_1A^2z_1 + \cdots + c_{n-1}A^nz_1$. Since $\phi(A) = 0$,

$\Rightarrow A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I = 0$

$\Rightarrow A^n = -(a_{n-1} + \cdots + a_0I)$

$$\Rightarrow B = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix}$$

Thus $dim(X) = n$. $\blacksquare$