進階代數(上) 第二次作業

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We always assume that a ring R has the multiplication identity 1.

1. (林家銘) Let A, B be ideals of a ring R. Show that AB is an ideal of R, where
$$AB := \{a_1b_1 + a_2b_2 + \cdots + a_nb_n \mid a_i \in A, b_i \in B\}.$$

2. (林育生) Let R be a commutative ring and P be an ideal of R. Show that P is prime if and only if the quotient ring R/P is an integral domain.

3. Let M be an ideal of a ring R. M is said to be *maximal* if $M \neq R$ and for any ideals N, we have
$$M \subseteq N \subseteq R \Longrightarrow N = M \text{ or } N = R.$$
   (a) (陳建文) Show that if M is a maximal ideal and R is commutative then the quotient ring R/M is a field.
   (b) (羅健峰) Show that if the quotient ring R/M is a division ring, then M is maximal.
   (c) (何昕暘) Give an example of a maximal ideal M in a ring R such that R/M is not a division ring.

4. (a) (賴德展) Let I_1, I_2, I_3 be ideals of a ring R such that $I_i + (I_j \cap I_k) = R$ for $\{i, j, k\} = \{1, 2, 3\}$. Show that for any $a_1, a_2, a_3 \in R$ there exists $a \in R$ such that $a - a_i \in I_i$ for all $i \in \{1, 2, 3\}$. (Hint. For each a_i there exists $b_i \in I_i$ and $c_i \in I_j \cap I_k$ such that $b_i + c_i = a_i$.)
   (b) (洪湧昇) Let I_1, I_2, I_3 be ideals of a ring R such that $I_i + (I_j \cap I_k) = R$ for $\{i, j, k\} = \{1, 2, 3\}$. Show that $R/(I_1 \cap I_2 \cap I_3)$ is isomorphic to direct product $R/I_1 \times R/I_2 \times R/I_3 := \{(\overline{a_1}, \overline{a_2}, \overline{a_3}) \mid \overline{a_i} \in R/I_i\}$ of rings R/I_1, R/I_2 and R/I_3 with addition and multiplication are defined componentwise.
   (c) (林志峰) Let n_1, n_2, n_3 be positive integers such that any two of them are relative primes. Show that for any integers a_1, a_2, a_3 there exists an integer a such that $a \equiv a_i \pmod{n_i}$ for $i = 1, 2, 3$. (這是中國餘式定理)
   (d) (呂融昇) Let n_1, n_2, n_3 be positive integers such that any two of them are relative primes. Show that $\mathbb{Z}/(lcm(n_1, n_2, n_3))$ is isomorphic to $\mathbb{Z}/(n_1) \times \mathbb{Z}/(n_2) \times \mathbb{Z}/(n_3)$.

5. An ideal J is *radical* of R if $a^n \in J$ for some positive integer n implies $a \in J$. Let I be an ideal of a commutative ring R and set
$$\sqrt{I} := \{a \in R \mid a^n \in I \text{ for some positive integer } n\}.$$
   (a) (連敏筠) Find $\sqrt{(24)}$ in $\mathbb{Z}$.
   (b) (施智懷) Show that $\sqrt{I}$ is a radical ideal of R.