進階代數(上) 第三次作業

上課老師: 翁志文

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1. Let $z = a + bi \in \mathbb{C}$ where $a, b \in \mathbb{R}$, and $N(z) := a^2 + b^2$ is the *norm* of z. Let $\mathbb{Z}[\sqrt{-d}] := \{a + b\sqrt{-d} \mid a, b \in \mathbb{Z}\}$, where $d \in \mathbb{N}$.
   (a) (邱鈺傑) Show $N(zz') = Z(z)N(z')$ for $z, z' \in \mathbb{C}$.
   (b) (蕭雯華) For $z \in \mathbb{Z}[\sqrt{-d}]$ show that $N(z)$ is a nonnegative integer.
   (c) (裴若宇) Show that the element $z \in \mathbb{Z}[\sqrt{-d}]$ is a unit if and only if $N(z) = 1$.
   (d) (林逸軒) Let $N(z)$ be a prime integer. Show hat z is irreducible in $\mathbb{Z}[\sqrt{-d}]$.
   (e) (陳巧玲) Find all units of $\mathbb{Z}[\sqrt{-5}]$.
   (f) (李光祥) Show that 3 is irreducible in $\mathbb{Z}[\sqrt{-5}]$.
   (g) (林詒琪) Show that 3 is not a prime in $\mathbb{Z}[\sqrt{-5}]$.

2. Let S be a nonempty subset of a commutative ring R. An element $d \in R$ is said to be a *greatest common divisor* of X if (i) $d|a$ for all $a \in S$; (ii) If $c|a$ for all $a \in S$, then $c|d$. The *least common multiple* of X can be defined similarly.
   (a) (葉彬) Find the greatest common divisor of 2 and $1 + \sqrt{-5}$ in $\mathbb{Z}[\sqrt{-5}]$.
   (b) (林家銘) Find the greatest common divisor of $6 - 6\sqrt{-5}$ and 18 in $\mathbb{Z}[\sqrt{-5}]$.

3. An commutative integral domain D is a *Euclidean domain* if there is a function $\mu : D \setminus \{0\} \to \mathbb{N}$ such that for all $a, b \in D$ with $b \neq 0$, there exist $q, r \in D$ such that $a = bq + r$, where $r = 0$ or $\mu(r) < \mu(b)$.
   (a) (林育生) Show that $\mathbb{Z}$ is a Euclidean domain.
   (b) (陳建文) Show that every Euclidean domain is a principal ideal domain.
   (c) (羅健峰) Show that the ring $\mathbb{Z}[\sqrt{-1}]$ is a Euclidean domain.
   (d) (何昕暘) Find all units of $\mathbb{Z}[\sqrt{-1}]$.
   (e) (賴德展) Determine all the prime elements in $\mathbb{Z}[\sqrt{-1}]$.
   (f) (洪湧昇) Find the least common multiple and the greatest common divisor of $11+3i$ and $8 - i$ in $\mathbb{Z}[i]$.