進階代數(上) 第四次作業

上課老師: 翁志文

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1. (林志峰) Determine the complete ring of quotients of the ring $\mathbb{Z}_n$ for each $n \geq 2$.
2. (呂融昇) Let R be an integral domain with quotient field F. Let T be an integral domain such that $R \subseteq T \subseteq F$. Show that the quotient field of T is isomorphic to F.
3. Let S be a multiplicative subset of an integral domain R such that $0 \notin S$.
   (a) (羅元勳) Show that if R is an integral domain, then so is $S^{-1}R$.
   (b) (連敏筠) Show that if R is a unique factorization domain, then so is $S^{-1}R$.
4. (施智懷) Let S be a multiplicative subset of a commutative ring R. Show that for each prime ideal P' of $S^{-1}R$ there exists a prime ideal P of R such that $P' = S^{-1}P$.
5. Let P be a prime ideal in a commutative ring R.
   (a) (邱鈺傑) Show that there is a one-to-one correspondence between the set of prime ideals Q which are contained in P and the set of prime ideals of R_P, given by $Q \to Q_P$.
   (b) (裴若宇) Show that the ideal P_P in R_P is the unique maximal ideal of R_P.
   (c) (蕭雯華) Give a ring R with two distinct prime ideals P, Q such that $Q \subseteq P$.
6. (林逸軒) Let M be a maximal ideal in a commutative ring R with identity and n be a positive integer. Show that the quotient ring R/M^n has a unique prime ideal and therefore is local.
7. (陳巧玲) Let R be a commutative ring with identity. Show that R is local if and only if for all $r, s \in R$,
$$r + s = 1 \Rightarrow r \text{ or } s \text{ is a unit.}$$
8. Let R be a commutative ring with identity. Consider the following three statements.

(i) R has a unique prime ideal.

(ii) Every nonunit is nilpotent.

(iii) R has a minimal prime ideal which contains all zero divisors, and all nonunits of R are zero divisors.

   (a) (李光祥) Find a ring R, which is not a field, satisfying the above conditions (i)-(iii).
   (b) (林詒琪) Show (ii)⇒(i).
   (c) (葉彬) Show (i)⇒(iii).
   (d) (林家銘) Show (iii)⇒(ii).