8

(a) $R = \mathbb{Z}_4 = \{0, 1, 2, 3\}$

unit:1,3
nonunit:0,2
nilpotent:0,2
zero division:2

(i) $P \subsetneqq R$
{0},{0,2}

(ii) ✓

(iii) ✓

(b) M : nonunit is nilpotent.
M is maximal ideal of R.
$a, b \in M, (a-b)^n \in M$.
$(ra)^n = 0 = (ar)^n, r \in R$.
M is ideal.
$M \subsetneqq N \subsetneqq R$.
$M \neq N$.
$N = R$.
$\exists a \in unit \in R$.
M is maximal ideal.
Let P be a prime ideal of R.
$P \subsetneqq M$.
Show $M \subsetneqq P$.
$a \in M$.
$a^n = 0$.
$aa^{n-1} = 0$.

(i) $a \in P$,done.

(ii) $a^{n-1} \in P$.

$$\begin{array}{l} \shortparallel \\ a \cdot a^{n-2} \\ \vdots \\ a \in P \\ \Rightarrow M \subsetneqq P. \end{array}$$

(c) Let P be the unique prime ideal.
$P \mapsto P_P$ is (unique) maximal ideal in R_P.
$\therefore {}^{R_P}\diagup_{P_P}$ is a field.
Suppose $a \in R$ is a zero divisor.
$\therefore \frac{a}{1}$ is a zero divisor in R_P.
$\Rightarrow \frac{a}{1} \in P_P = \{\frac{r}{t} | r \in P, t\ not\ inP\}$.
$\Rightarrow (at - r)s = 0$ for some $r \in P, t\ not\ inP_1, S \in R - P$.
$\Rightarrow ats = rs \in P$.
$\Rightarrow a \in P$.
Since P is prime.

$(i) \Rightarrow (ii)$ P is maximal prime ideal if

(1)P is a prime ideal.

(2)If P' is a prime ideal and $P' \subseteqq P$,then $P = P'$.

$(iii) \Rightarrow (ii)$ Let P be min prime ideal in R.

Let $\mathbb{Z}$ be the set of zero divisor.

Let N be the set of nonunits.

By (iii),$\mathbb{Z} \subseteqq P$ and $N \subseteqq \mathbb{Z}$.

Hence $N = \mathbb{Z}$.

Since $\mathbb{Z}$ is prime, $\mathbb{Z} = P = N$.

Hence R is local.

$C = \{I | I \ an \ ideal \ in \ R, \ with \ I \cap S = \varnothing\}$.

Take a "maximal" element Q in C.

Then Q is a prime ideal and $Q \neq R$.

Hence $P = Q \ \ \rightarrow\leftarrow$ since $Q \cap S = \varnothing$.

$P \cap S = \varnothing$.

Then $0 \in S$.

(d) $P :$ min prime ideal $\subseteqq R$.

$\mathbb{Z} :$ all zero divisor for R.

$\mathbb{Z} \subseteqq R$.

$N :$ all nonunit ,$N \nsubseteqq \mathbb{Z}$.

prove $x \in N, x^n = 0$ for some n.

$P \supseteqq N \Rightarrow P$ is a maxi ideal.

$\because P$ is min prime ideal.

$\therefore P$ is unique prime ideal.

$x \in N$.

$x^n \neq 0 \forall n$.

$S = \{x^n | n = 1, 2, \ldots\}$.

C is a collection.

$C = \{I \lhd R : I \cup S = \varnothing\}$.

$(C, \subseteqq)$ partial order set.

Take a maximal chain in C and let Q be the union of the chain.

Note $Q \in C$.

$a, b \in R, ab \in Q$.

Suppose $a \ not \ inQ, b \ not \ inQ$.

$a \in Q + (a), b \in Q + (b)$.

But Q is max in C$\Rightarrow$ $Q + (a) \cap S$.

$\Rightarrow \exists x^i \in Q + (a) \cap S, x^i \in Q + (b) \cap S$.

$\Rightarrow x^{i+h} \in (Q + (a))(Q + (b)) \subseteqq Q + (ab) = Q \rightarrow\leftarrow$

$\Rightarrow$Q is a prime ideal.

But $Q = P \rightarrow\leftarrow$

$\Rightarrow x^n = 0$ for some n.