進階代數(上) 第五次作業

上課老師: 翁志文

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1. (林育生)Let F be a field. Show that $F[x]$ is a Euclidean domain, a principal ideal domain and a unique factorization domain. (Hint. Theorem 6.2)

2. (a) (陳建文) Show that the polynomial $x+1$ is a unit in $\mathbb{Z}[[x]]$, but not a unit in $\mathbb{Z}[x]$.
   (b) (羅健峰) Show that x^2+3x+2 is irreducible in $\mathbb{Z}[[x]]$, but not in $\mathbb{Z}[x]$.

3. Let F be a field.
   (a) (何昕暘) Show that (x) is a maximal ideal in $F[x]$, but it is not the only maximal ideal.
   (b) (賴德展) Show that $F[[x]]$ is a principal ideal domain whose only ideals are 0, (1), (x^i) for $i \in \mathbb{N}$.

4. (a) (洪湧昇) Show that if $1-ab$ is a unit in a ring then so is $1-ba$.
   (b) (呂融昇) Prove that if an element of a ring has more than one right inverse then it has infinitely many. (Hint. Try $b+1-ba$.)
   (c) (林志峰) Let a, b be elements of a ring such that a, b and $ab-1$ are units. Show that $a-b^{-1}$ and $(a-b^{-1})^{-1}-a^{-1}$. (Hint. Try $aba-a$.)

5. Define the Möbius function $\mu(n)$ of positive integers by the following rules: (a) $\mu(1)=1$, (b) $\mu(n)=0$ if n has a square factor, (c) $\mu(n)=(-1)^s$ if $n=p_1p_2\cdots p_s$, where p_i are distinct primes.
   (a) (羅元勳) Prove that $\mu(n_1n_2)=\mu(n_1)\mu(n_2)$ if $gcd(n_1,n_2)=1$.
   (b) (連敏筠) Prove that
   $$\sum_{d|n}\mu(d)=\begin{cases}1, & \text{if } n=1;\\ 0, & \text{if } n\neq 1.\end{cases}$$
   (c) (施智懷) Let $g:\mathbb{N}\to R$ be a function and
   $$g(n)=\sum_{d|n}f(d).$$
   Show
   $$f(n)=\sum_{d|n}\mu(\frac{n}{d})g(d).$$
   (d) (邱鈺傑) Let F_q be a field with q elements. Show that the number of irreducible monic quadratic polynomials in $F[x]$ is $q(q-1)/2$.
   (e) (裴若宇) Let F_q be a field with q elements. Show that the number of irreducible monic cubic polynomials in $F[x]$ is $q(q^2-1)/3$.