進階代數(上) 第六次作業

上課老師: 翁志文

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1. (蕭雯華) Let F be a field and $f, g \in F[x]$ with degree of g at least 1. Show that there exist unique polynomials $f_0, f_1, \ldots, f_r \in F[x]$ with $\deg f_i < \deg g$ for all i and
$$f = f_0 + f_1 g + f_2 g^2 + \cdots + f_r g^r.$$

2. For $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x]$, $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}$ is called *the formal derivative* of $f(x)$.
   (a) (林逸軒) Suppose char $R = 0$ and $f(x) \in R[x]$ has positive degree. Show $f'(x) \neq 0$.
   (b) (陳巧玲) Suppose char $R = p \neq 0$ and $f(x) \in R[x]$. Show that $f'(x) = 0$ if and only if $f(x) = b_0 + b_1 x^p + b_2 x^{2p} + \cdots + b_n x^{np}$ for some $b_i \in R$.

3. (a) (李光祥) Let $c \in F$, where F is a field of characteristic $p \neq 0$. Show that $x^p - x - c$ is irreducible in $F[x]$ if and only if $x^p - x - c$ has no root in F.
   (b) (林詒琪) Show that $f(x) = x^5 - x - 1$ has no root in $\mathbb{Q}$.
   (c) (葉彬) Show that $f(x) = x^5 - x - 1$ is irreducible over $\mathbb{Q}$.
   (d) (葉彬提供) Show that $f(x) = x^5 - x - 15$ is reducible over $\mathbb{Q}$.

4. (a) (林家銘) Let D be an integral domain and $c \in D$. Show that $f(x)$ is irreducible in $D[x]$ if and only if $f(x - c)$ is irreducible.
   (b) (林育生) For each prime p show that the *cyclotomic polynomial* $f(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible in $\mathbb{Z}[x]$.

5. (a) (陳建文) If $a_0, a_1, \ldots, a_n$ are distinct elements of an integral domain D and $d_0, d_2, \ldots, d_n$ are any elements of D. Show that there is at most one polynomial $f(x)$ of degree at most n in $D[x]$ such that $f(a_i) = d_i$ for $0 \leq i \leq n$.
   (b) (羅健峰) If $a_0, a_1, \ldots, a_n$ are distinct elements of a field F and $d_0, d_2, \ldots, d_n$ are any elements of F. Show that
$$f(x) = \sum_{i=0}^{n} \frac{(x - a_0) \cdots (x - a_{i-1})(x - a_{i+1}) \cdots (x - a_n)}{(a_i - a_0) \cdots (a_i - a_{i-1})(a_i - a_{i+1}) \cdots (a_i - a_n)} d_i$$
   is the unique polynomial in $F[x]$ of degree at most n such that $f(a_i) = d_i$ for all i.