1. *Proof*:

   $(\Rightarrow)$

   $\forall u \in W$

   Since W is a subspace of $\mathbb{R}^n$, $\forall f \in \mathbb{R}[\lambda]$, $Au = \lambda u \in w$

   $(\Leftarrow)$

   Since $AW \subseteqq W$, $\forall f \in \mathbb{R}[\lambda]$, $f(\lambda)u = f(A)u \in W$.

2. *Proof*:

   Define : $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_2 \times \mathbb{Z}_6$ by $\phi(1,1) = \overline{(1,0)}$, $\phi(1,0) = \overline{(0,1)}$

   Since $(1,1), (1,0)$ and $\overline{(1,0)}, \overline{(0,1)}$ are those largest generate set of $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}_2 \times \mathbb{Z}_6$, respectly, this homomorphism is decided .

   And note that the kernel of ϕ is $((2,2),(6,0)) = ((2,2),(4,-2))$

   Thus , by the First Isomorphic Theorem , the result follows.

3. *Proof*:

   Since $M = M_1 + \cdots + M_n$, $\forall m \in M$, $\exists m_i \in M_i$, s.t. $m = m_1 + \cdots + m_n$

   Define : $\phi : M \to M_1 \oplus \cdots \oplus M_n$, by $\phi(m) = (m_1, \cdots, m_n)$

   (a) **Well − define**

   Suppose $\phi(m) = (m_1, \cdots, m_n) = (k_1, \cdots, k_n)$

   By definition , $m = m_1 + \cdots + m_n = k_1 + \cdots + k_n$

   $$\begin{aligned} \Rightarrow m_1 - k_1 &= (k_2 + \cdots + k_n) - (m_2 + \cdots + m_n) \\ &= (k_2 - m_2) + \cdots + (k_n - m_n) \end{aligned}$$

   This implies $M_2 + \cdots + M_n = 0$ and we have well-define.

   (b) **OnetoOne**

   Let $\phi(m) = (m_1, \cdots, m_n) = \phi(k)$

   then $m = m_1 + \cdots + m_n = k$

   (c) **Onto**

   $M = M_1 + \cdots + M_n$, $\forall m_i \in M_i$ $\exists m \in M$ s.t. $m_1 + \cdots + m_n = m$

   And ϕ is an isomorphism from M to $M_1 \oplus \cdots \oplus M_n$.

4. *Proof*:

   First I should prove that $N_1 \oplus N_2$ is a submodule of $M_1 \oplus M_2$, and this just follows from the fact that $N_1 \oplus N_2$ is a subgroup of $M_1 \oplus M_2$.

   And then , Let $\phi : M_1 \oplus M_2 \to M_1/N_1 \oplus M_2/N_1$ by $\phi(m_1, m_2) = (\overline{m_1}, \overline{m_2})$ is a homomorphism

   If we let $(x_1, x_2) \in Ker(\phi)$, $\phi(x_1, x_2) = (\overline{e_1}, \overline{e_2})$where $x_i \in M_i$ and e_i is identity of M_i , $(x_i - e_i) \in N_i$

   This implies $x_i \in N_i$ for i is 1,2.And it is clearly that Ker$\phi \supseteq N_1 \oplus N_2$

   Therefore , Kerϕ is $N_1 \oplus N_2$, and by First Isomorphic Theorem , the result follows.

5. $Proof$:

First I claim that g is one-to-one and f is onto

$\forall x, y \in N$ such that $g(x) = g(y)$

$\Rightarrow f \circ g(x) = f \circ g(y) \Rightarrow x = y$

This implies g is one-to-one.

And $\forall x \in N, \exists y \in M, where y = g(x) \Rightarrow f(y) = f \circ g(x) = x$ implies f is onto

Then $\forall a \in \text{Ker}f \cap \text{Im}g$, $\exists b\ inN$ s.t. $a = g(b)$ and $f(a) = 0$

$\Rightarrow f(a) = f \circ g(b) = 0 = b$

$\Rightarrow g(0) = a = 0$ implies $\text{Ker}f \cap \text{Im}g = 0$

Moreover , $\forall m \in M$ Let $n = g \circ f(n)$

Note that $n \in \text{Im}g$,

and $f(m - n) = f(m) - f(n)$

$= f(m) - f \circ g \circ f \circ (m)$

$= f(m) - (f \circ g) \circ f(m) = f(m) - f(m) = 0$

thus , $m - n \in \text{Ker}f \Rightarrow m \in \text{Ker}f + \text{Im}g$

Final the result follows.

6. P is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -12 & -2 & 1 & 0 \\ -3798 & -633 & 319 & -1 \end{bmatrix}$$

Q is

$$\begin{bmatrix} 0 & 0 & 14 & -27 \\ 0 & -1 & -39 & 95 \\ 0 & 1 & -2 & 4 \\ 1 & 7 & 81 & -161 \end{bmatrix}$$

PAQ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 610 \end{bmatrix}$$

7. PAQ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda - 1 & 0 \\ 0 & 0 & 0 & (\lambda - 1)^2(\lambda + 1) \end{bmatrix}$$