進階代數(上) 第八次作業

上課老師: 翁志文

2008 年十一月二十日

1. (a) (施智懷) Let F be a field and $f(x) \in F[x]$ with positive degree. Show that there exists a field E with $F \subseteq E$ and an element $a \in E$ such that $f(a) = 0$.

   (b) (邱鈺傑) Let F be a field. Show that there exists a field E with $F \subseteq E$ such that for any $f(x) \in F[x]$ of positive degree there exists an element $a \in E$ such that $f(a) = 0$.

   (c) (斐若宇) F be a field. Show that there exists a field E with $F \subseteq E$ such that for any $f(x) \in E[x]$ of positive degree there exists an element $a \in E$ such that $f(a) = 0$.

2. (蕭雯華) Let A be an $n \times n$ invertible matrix over a PID D. Show that A is a product of elementary matrices of types I, II, III and extra type.

3. (林逸軒) Determine the structure of $\mathbb{Z}$-module $\mathbb{Z}^3/K$ where K is generated by $f_1 = (2, 1, -3)$, $f_2 = (1, -1, 2)$.

4. (陳巧玲) Let $D = \mathbb{Z}[\sqrt{-1}]$. Determine the structure of D^3/K where K is generated by $f_1 = (1, 3, 6)$, $f_2 = (2 + 3i, -3i, 12 - 18i)$, $f_3 = (2 - 3i, 6 + 9i, -18i)$, $i = \sqrt{-1}$. What is the cardinality of D^3/K?

5. (李光祥) Show that any finite generated module M over a PID is a direct sum of its torsion submodule M_t and a free submodule. (Hint. Use structure theorem)

6. Let M be a finitely generated torsion module over a PID D.

   (a) (葉彬) Show that there exists a unique finite set of primes $p_1, p_2, \ldots, p_h$ in D such that
   $$M = M_{p_1} \oplus M_{p_2} \oplus \cdots \oplus M_{p_h},$$
   where $M_{p_i} = \{m \in M \mid p_i^k m = 0 \text{ for some } k \in \mathbb{N}\}$. (Hint. p_i appears in the generators of order ideals of generators of M.)

   (b) (林家銘) Show that for each $1 \leq i \leq h$, $M_{p_i} = Dz_{i1} \oplus Dz_{i2} \oplus \cdots \oplus Dz_{ir_i}$ with
   $$O_{z_{i1}} \supseteq O_{z_{i2}} \supseteq \cdots \supseteq O_{z_{ir_i}} \neq 0$$
   for some $r_i \in \mathbb{N}$ and some $z_{ij} \in M_{p_i}$.

   (c) (林育生) Let G be a finite abelian group with operation $+$. Show that G is a finitely generated torsion $\mathbb{Z}$-module under the natural definition of ng for $n \in \mathbb{Z}$ and $g \in G$.

   (d) (陳建文)Determine all the finite abelian groups of order 72 up to isomorphism.