進階代數(上) 第九次作業

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1. Suppose
$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}.$$
   (a) (林詒琪) Find the rational canonical form of A and a 4×4 invertible matrix S such that $S^{-1}AS$ is in rational canonical form.
   (b) (羅健峰) Find the Jordan canonical form of A and a 4×4 invertible matrix U such that $U^{-1}AU$ is in Jordan canonical form.

2. (何昕暘) Prove that two $n \times n$ matrices A, B over $\mathbb{R}$ are similar if and only if the matrices $\lambda I - A$, $\lambda I - B$ are equivalent over $\mathbb{R}[\lambda]$.

3. (賴德展) Prove that any matrix is similar to its transpose.

4. (洪湧昇) Let A ba an $n \times n$ matrix. Show that the $\mathbb{R}[\lambda]$-module $\mathbb{R}^n$ determined by A is cyclic if and only if the characteristic polynomial $f(\lambda)$ of A is the minimum polynomial of A.

5. (林志峰) Show that the following $p \times p$ matrices over $\mathbb{Z}_p$, p a prime, are similar:
$$\begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ 0 & \cdot & & 0 & 1 \\ 1 & 0 & & & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & & 0 \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ 0 & & & & 1 \end{pmatrix}.$$

6. (呂融昇)Show that the $n \times n$ matrices A, B over $\mathbb{C}$ are similar if and only if for every $a \in \mathbb{C}$ and $k \in \mathbb{N}$
$$\operatorname{rank}(aI - A)^k = \operatorname{rank}(aI - B)^k.$$

7. (羅元勳) Show that any matrix over $\mathbb{R}$ is similar to a matrix consisting of diagonal blocks which have one of the following forms:
$$\begin{pmatrix} r & & & & 0 \\ 1 & r & & & \\ & 1 & \ddots & & \\ & & \ddots & r & \\ 0 & & & 1 & r \end{pmatrix}, \begin{pmatrix} \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix} & & & & & 0 \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix} & & & & \\ & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \ddots & & & \\ & & \ddots & \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix} & & \\ 0 & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix} & \end{pmatrix},$$
where $a^2 - 4b < 0$.