

Skew-Symmetric power

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$S_k \stackrel{\text{def}}{=} \{\pi | \pi : \{1, \dots, k\} \rightarrow \{1, \dots, k\} \text{ is a bijection}\}$
 $\text{sgn}(\pi) = (-1)^{\text{length}(\pi)}$ where length is the smallest number to write π as a product of transpositions.

Example 1.

$$\begin{aligned} \pi &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ &\implies \text{length}(\pi) = 2 \end{aligned}$$

We write $\bigotimes^k \mathbb{R}^n = \underbrace{\mathbb{R}^n \otimes \mathbb{R}^n \otimes \dots \otimes \mathbb{R}^n}_k$

and for $\pi \in S_k$. we write $\pi(x_1 \otimes x_2 \otimes \dots \otimes x_k) = x_{\pi(1)} \otimes x_{\pi(2)} \otimes \dots \otimes x_{\pi(k)}$.
 and generalize to all elements in $\bigotimes^k \mathbb{R}^n$.

Definition 1. $\bigwedge^k \mathbb{R}^n = \{x \in \bigotimes^k \mathbb{R}^n | \pi(x) = \text{sgn}(\pi)x \text{ for any } \pi \in S_k\}$.

Example 2. : Let $\{e_i\}$ be standard basis of \mathbb{R}^n .

1. $e_1 \otimes e_2 \notin \bigwedge^2 \mathbb{R}^2$ since for $\pi = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
 $\pi(e_1 \otimes e_2) = e_1 \otimes e_2 \neq \text{sgn}(\pi)e_1 \otimes e_2$.
2. $(e_1 \otimes e_2 - e_2 \otimes e_1) \in \bigwedge^2 \mathbb{R}^2$.
 $\text{Since } \pi(e_1 \otimes e_2 - e_2 \otimes e_1) = e_2 \otimes e_1 - e_1 \otimes e_2 = \text{sgn}(\pi)(e_1 \otimes e_2 - e_2 \otimes e_1)$

Note 1. :

1. For any $x_1, x_2, \dots, x_k \in \mathbb{R}^n$.

$$\sum_{\pi \in S_k} \text{sgn}(\pi) \pi(x_1 \otimes x_2 \otimes \dots \otimes x_k) \in \bigwedge^k \mathbb{R}^n.$$

2. $x_1 \wedge x_2 \wedge \dots \wedge x_k \stackrel{\text{def}}{=} \sum_{\pi \in S_k} \text{sgn}(\pi) \pi(x_1 \otimes x_2 \otimes \dots \otimes x_k)$.
3. $(x_1 + x'_1) \wedge x_2 \wedge \dots \wedge x_k = (x_1 \wedge x_2 \wedge \dots \wedge x_k) + (x'_1 \wedge x_2 \wedge \dots \wedge x_k)$.
4. $x_1 \wedge x_1 \wedge x_3 \wedge x_4 \wedge \dots \wedge x_k = 0$.

Fact:

For $\alpha = \{\alpha_1 < \alpha_2 < \dots < \alpha_k | \alpha_i \in \{1, 2, \dots, n\}\}$.

define $e_\alpha \stackrel{\text{def}}{=} e_{\alpha 1} \wedge e_{\alpha 2} \wedge \dots \wedge e_{\alpha k}$

Then $\{e_\alpha | \alpha \subseteq \{1, 2, \dots, n\}, |\alpha| = k\}$ is a basis of $\bigwedge^k \mathbb{R}^n$.

In particular, $\bigwedge^k \mathbb{R}^n$ has dimension $\binom{n}{k}$.

Definition 2. For $\mathcal{A} \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^s)$. define $\bigwedge^k \mathcal{A} \in \text{Hom}_{\mathbb{R}}(\bigwedge^k \mathbb{R}^n, \bigwedge^k \mathbb{R}^s)$.

by $\bigwedge^k \mathcal{A}(e_{\alpha 1} \wedge e_{\alpha 2} \wedge \dots \wedge e_{\alpha k}) = (\mathcal{A}e_{\alpha 1}) \wedge (\mathcal{A}e_{\alpha 2}) \wedge \dots \wedge (\mathcal{A}e_{\alpha k})$.

Note 2. :

$$1. \bigwedge^k (\mathcal{A}\mathcal{C}) = (\bigwedge^k \mathcal{A})(\bigwedge^k \mathcal{C})$$

$$2. (\bigwedge^k \mathcal{A})^{-1} = \bigwedge^k \mathcal{A}^{-1}$$

if the size of \mathcal{C} is correct and \mathcal{A} is invertible in 2.

Fact:

$(\bigwedge^k \mathcal{A})e_\beta = \sum_{\substack{\alpha \in \{1, \dots, s\} \\ |\alpha|=k}} (\det \mathcal{A}[\alpha|\beta]) e_\alpha$ where $\mathcal{A}[\alpha|\beta]$ is a $k \times k$ submatrix of \mathcal{A} .

with rows in α and columns in β remaining, and $\beta \subseteq \{1, 2, \dots, n\}$ of size $|\beta| = k$.

i.e $\bigwedge^k \mathcal{A}$ is a matrix of size $\binom{s}{k} \times \binom{n}{k}$ with α, β entries $\det \mathcal{A}[\alpha|\beta]$.