Homework 11

1. Determine all the $n \times n$ matrices in rational canonical form over $\mathbb{R}$ with minimal polynomial $\lambda^2 + 1$.

Sol. Let A be such an $n \times n$ matrix. Consider $\lambda I - A^t$ as its rational canonical form. Then $\det(\lambda I - A^t)=\det(diag(1, \ldots, 1, d_1, \ldots, d_s)) = d_1 d_2 \cdots d_s$. Since $d_s = \lambda^2 + 1$, we have $d_1 = \cdots = d_s = \lambda^2 + 1$, $s = \frac{n}{2}$, n is even. Hence the canonical form is like:

$$\Lambda = \begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & & & & \\ & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}_{n \times n}$$

$\square$

2. Prove the injective case in The Short Five Lemma.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r, "g"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

Proof. Assume α, γ are both injective. Let $b \in B$ and suppose $\beta(b) = 0$. By commutative, $rg(b) = g'\beta(b) = g'(0) = 0$. Since γ is injective, we have $g(b) = 0$. Then by the definition, $b \in Ker(g) = Im(f)$, and hence $b = f(a)$, for some $a \in A$. By commutative again, $f'\alpha(a) = \beta f(a) = \beta(b) = 0$. Then $\alpha(a) \in Ker(f')$. Since f' is an 1-1 mapping, $\alpha(a) = 0$. And furthermore, $a = 0$ because of the injection of α. Thus we have $b = f(a) = 0$ and β is injective. $\square$

3. (The Five Lemma) Suppose we have the following commutative diagram of R-module homomorphisms with exact rows.

$$
\begin{tikzcd}
M_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\circlearrowleft"] & M_2 \arrow[r, "g_2"] \arrow[d, "f_2"'] \arrow[dr, phantom, "\circlearrowleft"] & M_3 \arrow[r, "g_3"] \arrow[d, "f_3"'] \arrow[dr, phantom, "\circlearrowleft"] & M_4 \arrow[r, "g_4"] \arrow[d, "f_4"'] \arrow[dr, phantom, "\circlearrowleft"] & M_5 \arrow[d, "f_5"'] \\
N_1 \arrow[r, "h_1"'] & N_2 \arrow[r, "h_2"'] & N_3 \arrow[r, "h_3"'] & N_4 \arrow[r, "h_4"'] & N_5
\end{tikzcd}
$$

(a) Prove directly f_1 surjection and f_2, f_4 injection $\Rightarrow$ f_3 injection.

Proof. Pick any $x \in M_3$ such that $f_3(x) = 0$. Then $0 = h_3(f_3(x)) = f_4(g_3(x))$. Since f_4 is 1-1, we have $g_3(x) = 0$. In the other way, there exists an element a in M_2 such that $g_2(a) = x$. Then $h_2(f_2(a)) = f_3(g_2(a)) = f_3(x) = 0$. Let $b \in N_1$, $c \in M_1$ such that $h_1(b) = f_2(a)$ and $f_(c) = b$. Thus, $f_2(a) = h_1(b) = h_1(f_1(c)) = f_2(g_1(c))$. By the injection of f_2, $a = g_1(c)$. This implies $x = g_2(a) = g_2(g_1(c)) = 0$. $\square$

(b) Prove (a) by using The Short Five Lemma.

Proof. Consider the following diagram obtained from above one:

$$
\begin{tikzcd}
0 \arrow[r] & M_2/Ker(g_2) \arrow[r, "\overline{g_2}"] \arrow[d, "\overline{f_2}"'] \arrow[dr, phantom, "\circlearrowleft"] & M_3 \arrow[r, "\overline{g_3}"] \arrow[d, "f_3"'] \arrow[dr, phantom, "\circlearrowleft"] & Img(g_3) \arrow[r] \arrow[d, "\overline{f_4}"'] & 0 \\
0 \arrow[r] & N_2/Ker(h_2) \arrow[r, "\overline{h_2}"'] & N_3 \arrow[r, "\overline{h_3}"'] & Img(h_3) \arrow[r] & 0
\end{tikzcd}
$$

We note that $\overline{f_2} := M_2/Ker(g_2) \to N_2/Ker(h_2)$ by $\overline{m_2} \mapsto \overline{f_2(m_2)}$, and the definition of $\overline{f_4}$ is similar. Using The Short Five Lemma, it suffices to check $\overline{f_2}, \overline{f_4}$ are well-defined, injective and the diagram is commutative. Well-defined is routine, so we only show that $\overline{f_2}$ is injective and the commutative part.

First, pick any $\overline{m_2} \in M_2/Ker(g_2)$ such that $\overline{f_2(m_2)} = 0$. Since $f_2(m_2) \in Ker(h_2) = Img(h_1)$, $f_2(m_2) = h_1(n_1) = h_1(f_1(m_1)) = f_2(g_1(m_1))$, for some $m_1 \in M_1$ and $n_1 \in N_1$. Then $m_2 = g_1(m_1) \in Img(g_1) = Ker(g_2)$, implies that $\overline{m_2} = 0$. Hence $\overline{f_2}$ is injective.

Furthermore, $f_3(\overline{g_2}(\overline{m_2})) = f_3(g_2(m_2)) = h_2(f_2(m_2)) = \overline{h_2}(\overline{f_2(m_2)}) = \overline{h_2}(\overline{f_2}(\overline{m_2}))$. Then $f_3\overline{g_2} = \overline{h_2}\overline{f_2}$, and the diagram is commutative. And hence f_3 is injective by The Short Five Lemma. $\square$

(c) Prove directly f_5 injection and f_2, f_4 surjection $\Rightarrow$ f_3 surjection.

Proof. Pick any $y \in N_3$, there exists $a \in M_4$ such that $f_4(a) = h_3(y)$. Then $0 = h_4(h_3(y)) = h_4(f_4(a)) = f_5(g_4(a))$. This implies $g_4(a) = 0$ and thus $a \in Ker(g_4) = Img(g_3)$. Hence there exists $b \in M_3$ such that $g_3(b) = a$.

Now, the following is routine. $h_3(f_3(b) - y) = f_4(g_3(b) - h_3(y)) = 0$, then there exists $c \in N_2$ such that $h_2(c) = f_3(b) - y$. Since f_2 is onto, there exists $d \in M_2$ such that $f_2(d) = c$. Then $h_2(f_2(d)) = f_3(b) - y$. Thus, we have $f_3(g_2(d)) = f_3(b) - y$. Hence $y = f_3(b - g_2(d))$. $\square$

(d) Prove (c) by using The Short Five Lemma.

Proof. Consider the following diagram similar to (b):

$$\begin{tikzcd}
0 \arrow[r] & M_2/Ker(g_2) \arrow[r, "\overline{g_2}"] \arrow[d, "\overline{f_2}"'] \arrow[dr, phantom, "\circlearrowright"] & M_3 \arrow[r, "\overline{g_3}"] \arrow[d, "f_3"'] \arrow[dr, phantom, "\circlearrowright"] & Img(g_3) \arrow[r] \arrow[d, "\overline{f_4}"'] & 0 \\
0 \arrow[r] & N_2/Ker(h_2) \arrow[r, "\overline{h_2}"'] & N_3 \arrow[r, "\overline{h_3}"'] & Img(h_3) \arrow[r] & 0
\end{tikzcd}$$

We note that the definitions of all new functions are equal to those in (b). Similar to (b), it suffices to show that $\overline{f_2}, \overline{f_4}$ are well-defined, surjective and the diagram is commutative.

Pick $n_3 \in N_3$, we want to find $m_3 \in M_3$ such that $f_4(g_3(m_3)) = h_3(n_3)$. Since f_4 is onto, we can find $m_4 \in M_4$ such that $f_4(m_4) = h_3(n_3)$. Then $0 = h_4(h_3(n_3)) = h_4(f_4(m_4)) = f_5(g_4(m_4))$. And since f_5 is injective, we have $g_4(m_4) = 0$, i.e. $m_4 \in Ker(g_4) = Img(g_5)$. Choose $m_3 \in M_3$ such that $g_3(m_3) = m_4$. Then $f_4(g_3(m_3)) = f_4(m_4) = f_3(n_3)$. Finally, by the Short Five Lemma, f_3 is surjective. $\square$