4. Suppose $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$.

Then there exists $h: M_2 \to M$ s.t $gh = I_{M_2}$

$\Leftrightarrow 0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ and $0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0$ are isomorphic.

Pf: ($\Rightarrow$)

$$\begin{array}{ccccccc} 0 & \to & M_1 & \xrightarrow{f} & M & \underset{\rho}{\overset{\pi}{\rightleftarrows}} & M_2 \to 0 \\ & & I_{M_1}\downarrow & & \phi\downarrow & & \downarrow I_{M_2} \\ 0 & \to & M_1 & \xrightarrow{f} & M & \xrightarrow{g} & M_2 \to 0 \end{array}$$

Let $m_1 \in M_1, m_2 \in M_2$.
Define $\phi: M_1 \oplus M_2 \to M$ by $\phi: ((m_1, m_2)) = f(m_1) + h(m_2)$
Claim:commutative

$\phi\tau(m_1) = \phi((m,0)) = f(m_1) + h(0) = f(m_1) = fI_{M_1}(m_1)$

$g\phi((m_1, m_2)) = g(f(m_1) + h(m_2))$
$= gf(m_1) + gh(m_2)$
$= 0 + I_{M_2}(m_2)$
$= I_{M_2}\pi((m_1, m_2))$

By Five Short Lemma, ϕ is an isomorphism.
Thus these two exact sequences are isomorphic.
($\Leftarrow$)
Define $\rho: M_2 \to M_1 \oplus M_2$ by $\rho(m_2) = (0, m_2)$
Let $h = \phi\rho \Rightarrow gh = I_{M_2}$.

□

5. (a) Show that if $\oplus_{i\in I} P_i$ is projective then P_i is projective for each $i \in I$ (b) Show that if P_i for each $i \in I$ is projective then $\oplus_{i\in I} P_i$ is projective.

Pf: (a)

$\exists \ L$ s.t $L \oplus (\oplus_{i\in I} P_i)$ is free module.

$\Rightarrow (L \oplus (\oplus_{\substack{i\in I \\ i\neq j}} P_i)) \oplus P_j$ is free module. $\forall j \in I$

$\Rightarrow P_j$ is projective, $\forall j \in I$

(b)

$\because$ P_i is projective, $\forall i \in I$

$\therefore \exists \ L_i$ s.t $L_i \oplus P_i$ free, $\forall i \in I$

Consider $\oplus_{i\in I}(L_i \oplus P_i)$ is also free $\cong$ $(\oplus_{i\in I} L_i) \oplus (\oplus_{i\in I} P_i)$

$\therefore \oplus_{i\in I} P_i$ is projective.

□

6. Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

Pf:

(P is projective R-module $\Leftrightarrow \exists$ L is a R-module s.t $L \oplus P$ is free R-module)

Assume there exists L is a $\mathbb{Z}$-module s.t $L \oplus \mathbb{Q}$ is free $\mathbb{Z}$-module.

We try to get contradiction from this.

$\odot$ Finite Dimension

$\exists \quad (l_1, \frac{n_1}{m_1}), (l_2, \frac{n_2}{m_2}), ..., (l_k, \frac{n_k}{m_k}) \in L \oplus \mathbb{Q}$ is a basis of $L \oplus \mathbb{Q}$

where $l_i \in L$, $\frac{n_i}{m_i} \in \mathbb{Q}$, and $(n_i, m_i) = 1$.

take $a = \frac{1}{2m_1 m_2 \cdots m_k}$, then $(0, a) \in span((l_i, \frac{n_i}{m_i}), i = 1, ..., k)$

$\exists \quad c_1, c_2 \cdots c_k \in \mathbb{Z}$ s.t $(0, \frac{1}{2m_1 m_2 \cdots m_k}) = c_1(l_1, \frac{n_1}{m_1}) + c_2(l_2, \frac{n_2}{m_2}) + \cdots + c_k(l_k, \frac{n_k}{m_k})$

$$\Rightarrow \frac{1}{2m_1 m_2 \cdots m_k} = c_1 \frac{n_1}{m_1} + c_2 \frac{n_2}{m_2} + \cdots + c_k \frac{n_k}{m_k} = \frac{M}{m_1 m_2 \cdots m_k}$$

for some $M \in \mathbb{Z}$ $\rightarrow\leftarrow$

$\odot$ Infinite Dimension

$\exists S \subseteq L \oplus \mathbb{Q}$ is a basis and $| S | = \infty$, and $(0,1) \in span(S)$

$\exists \quad (l_1, \frac{n_1}{m_1}), (l_2, \frac{n_2}{m_2}), ..., (l_k, \frac{n_k}{m_k}) \in L \oplus \mathbb{Q}$ and $\exists \quad c_1, c_2 \cdots c_k \in \mathbb{Z}$

s.t $(0,1) = c_1(l_1, \frac{n_1}{m_1}) + c_2(l_2, \frac{n_2}{m_2}) + \cdots + c_k(l_k, \frac{n_k}{m_k})$,

take $a = \frac{1}{2m_1 m_2 \cdots m_k}$, $(0, a) \notin span((l_i, \frac{n_i}{m_i}), i = 1, ..., k)$,

but $a \in span(S)$, take $(l_{k+1}, \frac{n_{k+1}}{m_{k+1}}), (l_{k+2}, \frac{n_{k+2}}{m_{k+2}}), ..., (l_{k+b}, \frac{n_{k+b}}{m_{k+b}}) \in S$

s.t $(0, a) \in span((l_i, \frac{n_i}{m_i}), i = 1, ..., k+b)$

$\exists \quad c_1', c_2' \cdots c_k' \in \mathbb{Z}$ s.t $(0, a) = c_1'(l_1, \frac{n_1}{m_1}) + c_2'(l_2, \frac{n_2}{m_2}) + \cdots + c_{k+b}'(l_{k+b}, \frac{n_{k+b}}{m_{k+b}})$

$\exists \quad c_{k+b}' \neq 0$ for some i

$$2m_1 m_2 \cdots m_k \ (0, a) = (0,1) = 2m_1 m_2 \cdots m_k \ (c_1'(l_1, \frac{n_1}{m_1}) + c_2'(l_2, \frac{n_2}{m_2}) + \cdots + c_{k+b}'(l_{k+b}, \frac{n_{k+b}}{m_{k+b}}))$$

$\because \exists \ c_{k+b}' \neq 0 \Rightarrow \quad S$ isn't linearly independent.

□