1(a).

Since $Q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -(\lambda-1) \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & (\lambda-2)^2 \end{pmatrix}$

$$P = \begin{pmatrix} 0 & 1 & -(\lambda-3) & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -(\lambda-3)^2 & (\lambda-3)^2 & 1 \\ \end{pmatrix}$$

Then $P(\lambda I - A^t)Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda-2 & 0 \\ 0 & 0 & 0 & (\lambda-2)^2(\lambda-3) \end{pmatrix}$

Hence $\Lambda = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 \\ 0 & 1 & 0 & -16 \\ 0 & 0 & 1 & 7 \end{pmatrix}$

Now, we want to find S.

First, $Q^{-1} = \begin{pmatrix} 1 & 0 & -(\lambda-2)^2 & 1 \\ 0 & 1 & (\lambda-1) & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Then $\begin{pmatrix} v_1 \\ v_2 \\ z_1 \\ z_2 \end{pmatrix} = Q^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_1 - (\lambda-2)^2 u_3 + u_4 \\ u_2 + (\lambda-2)^2 v_3 \\ u_1 \\ u_3 \\ \end{pmatrix} \in R[\lambda]^4$

This implies

$\eta(z_1) = \eta(u_1) = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\eta(z_2) = \eta(u_3) = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$\lambda\eta(z_2) = A\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\1\\0\end{pmatrix}$$

$$\lambda^2\eta(z_2) = A^2\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\-4\\0\\1\end{pmatrix}.$$

So, $S = \begin{pmatrix}1 & 0 & 0 & 1\\0 & 0 & -1 & -4\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}$

$\implies S^{-1}AS = \Lambda$

1(b).

By (a), we get $\eta(z_1) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\eta(z_2) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$.

Moreover, $(A-2I)\eta(z_1) = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

So, $A(\eta(z_1)) = 2(\eta(z_1))$.

And $(A-2I)[(A-3I)]\eta(z_2)] = (A-2I)\begin{pmatrix}0\\-1\\-2\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

So, $A[(A-3I)]\eta(z_2)] = 2[(A-3I)]\eta(z_2)]$.

$\Rightarrow (A-2I)[(A-2I)(A-3I)\eta(z_2)] = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

However, this also implies

$$(A-3I)[(A-2I)(A-2I)\eta(z_2)] = (A-3I)[(A-2I)^2\eta(z_2)] = (A-3I)\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Thus, we get $U = \begin{pmatrix}1 & 0 & 1 & 1\\ 0 & -1 & 1 & 0\\ 0 & -2 & 1 & 0\\ 0 & 0 & 1 & 1\end{pmatrix}$ and $J = \begin{pmatrix}2 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 3\end{pmatrix}$.

2.

($\Rightarrow$)
A **and** B **are similar**
$\Rightarrow \exists S \in R^{n\times n}$ invertible, s.t $B = S^{-1}AS$
$\Rightarrow \lambda I - B = \lambda S^{-1}S - S^{-1}AS = S^{-1}(\lambda S) - S^{-1}(AS) = S^{-1}(\lambda I - A)S$
Since S and S^{-1} are invertible, $\lambda I - A$ and $\lambda I - B$ are equivalent.

($\Leftarrow$)
$\lambda I - A$ **and** $\lambda I - B$ **are equivalent**
$\Rightarrow \exists P, Q \in R^{n\times n}$ invertible, such that $\lambda I - B = P(\lambda I - A)Q$
$\Rightarrow \lambda I - B^t = (\lambda I - B)^t = (P(\lambda I - A)Q)^t = Q^t(\lambda I - A)^t P^t$
$\Rightarrow \lambda I - A^t$ and $\lambda I - B^t$ are equivalent
$\Rightarrow \lambda I - A^t$ and $\lambda I - B^t$ has the same Smith normal form $diag(d_1(\lambda), \dots d_n(\lambda))$
$\Rightarrow \exists S, T \in R^{n\times n}$ invertible, s.t $S^{-1}AS = \Lambda$ and $T^{-1}BT = \Lambda$
$\Rightarrow B = TS^{-1}AST^{-1} = (ST^{-1})^{-1}A(ST^{-1})$
$\Rightarrow A$, B are similar.

3.

Let the matrix be A
$P(\lambda I - A^t)Q = D$
$Q^t(\lambda I - A)^t P^t = D^t = D$
Therefore, $(\lambda I - A^t)$and $(\lambda I - A)$ has the same SNF.
$\Rightarrow (\lambda I - A^t)$ and $(\lambda I - A)$ have the same rational canonical form.
Then $\exists S$, T s.t $S^{-1}AS = \Lambda$ and $T^{-1}A^tT = \Lambda$.
$\Rightarrow A^t = (TS^{-1})A(ST^{-1}) = ((ST^{-1})^{-1})A(ST^{-1})$.
Therefore, A and A^t similar.

4.

($\Rightarrow$)

$R[\lambda]$-module R^n determind by A is cyclic $\Leftrightarrow R^n \cong \frac{D}{(d)}$

$$\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & d \end{pmatrix} \approx (\lambda I - A^t)$$

$\Rightarrow det(\lambda I - A^t$is d.

$\Rightarrow$The minimal polynomial of $A = d$

($\Leftarrow$)

Suppose it is not cyclic.

$$\text{Then } (\lambda I - A^t) \approx \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & d_1 & & \\ & & & & & \ddots & \\ 0 & & & & & & d_s \end{pmatrix}, s > 1.$$

$det(\lambda I - A^t) = d_1 \cdots d_s$

The minimal polynomial is d_s.

5.

$$A = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & 1 & \\ 1 & & & & 0 & \end{pmatrix} = S \begin{pmatrix} 0 & & & & 0 & 1 \\ 1 & 0 & & & & 0 \\ & 1 & 0 & & & \\ & & \ddots & \ddots & & \\ 0 & & & & 1 & 0 \end{pmatrix} S^{-1}.$$

$$(\lambda I - A^t) \cong \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \lambda^p - 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 & & & 0 \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & \ddots & 1 \\ 0 & & & & 1 \end{pmatrix} = T \begin{pmatrix} 1 & & & & 0 \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & 1 & \\ 0 & & & 1 & 1 \end{pmatrix} S^{-1}.$$

$$(\lambda I - B^t) \cong \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & (\lambda-1)^p \end{pmatrix}$$

Since $(\lambda^p - 1) = (\lambda - 1)^p, A \approx B$.

6.
First, we have to know that A and B are similar iff J_A=J_B.

$(\Rightarrow)$
Since, A and B similar, $\exists C$ is invertible, such that $CAC^{-1} = B$.
Moreover, $\exists$ P and Q are invertible, such that $PAP^{-1} = J_A$, $QBQ^{-1} = J_B$.
So, $P^k(aI_n - A)^k(P^{-1})^k = (aI_n - J_A)^k, Q^k(aI_n - B)^k(Q^{-1})^k = (aI_n - J_B)^k$
Thus, $rank(aI_n - A)^k = rank(aI_n - J_A)^k = rank(aI_n - J_B)^k = rank(aI_n - B)^k$.

$(\Leftarrow)$
Assume A, B have distinct Jordan form.
Then, we have two cases of $rank(\lambda I - B)^k$ concerning the eigenvalue of B.

(1)λ is not an eigenvalue of B:

$rank(\lambda I - A) = rank(\lambda I - J_A) < n$

$rank(\lambda I - B) = rank(\lambda I - J_B) = n$

So, $rank(\lambda I - A) \neq rank(\lambda I - B)$ (a contradiction!)

(2)A and B have different Jordan block form corresponding to λ :

For $rank(\lambda I - B)^k = n - kt - s, where \lambda$ is an eigenvalue with t Jordan blocks of size $\geq k$ with a in the diagonal and s is the sum of the size of blocks with $< k$.

Then since A and B have different Jordan block form corresponding to λ, there exists some k such that $rank(\lambda I - B)^k \neq rank(\lambda I - A)^k$ (a contradiction!)

7.

Let A be $2\ell \times 2\ell$ matrix:

$$A = \begin{pmatrix} \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix} & & & & & 0 \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix} & & & & \\ & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix} & & & \\ & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix} & & \\ & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix} & \\ 0 & & & & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$$

Let $f(\lambda) = \lambda^2 + a\lambda + b$

Then
(1)
$f(B) = O : \lambda I - B^t$ is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & -f(\lambda) \end{pmatrix}$
$f(\lambda)$is characteristic polynomial of B.

Suppose $f(\lambda) = (\lambda - \alpha)(\lambda - \beta)$ for $\alpha, \beta \in C$.

Then we have three cases to discuss:

case1:
$\alpha = \beta \in R$
B is similar to $\begin{pmatrix} \alpha & 0 \\ 1 & \beta \end{pmatrix}$

case2:
$\alpha \neq \beta \in R$
B is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

case3:
$\alpha = \bar{\beta} \in C \neq R$
B is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

In any cases $f(B) = O$.
This is another way to prove Caley-Hamilton Theorem.

(2)
Matrices multiplication can be in block form product.

(3)
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & bc & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ ab & 0 & 0 & 0 \\ 0 & bc & 0 & 0 \end{pmatrix}$$
If $a, b, c \neq 0$ invertible, then $ab, bc \neq 0$.

So, by (1)(2)(3), we have the following results:

(1)A has minimal polynomial $(f(\lambda))^\ell$.

(2)$(f(\lambda))^\ell$ is the characteristic polynomial.

(3) $\lambda I - A^t$ is equivalent to $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & (f(\lambda))^\ell \end{pmatrix}$