Advanced Algebra II Class Note

1.1 Semigroups, Monoids and Groups

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Definition

(1) A semigroup is a nonempty set $G$ with a binary operation on $G$
such that $a\,(bc) = (ab)\,c$ for $a, b, c \in G$.

(2) A monoid is a semigroup $G$ with an identity element $e$.
i.e. $ae = a = ea$ for all $a \in G$.

Note

If $G$ is a semigroup without identity $e$, then $G \cup \{e\}$ is a monoid, where the product $a \cdot e := a$ and $e \cdot a := a$ for all $a \in G \cup \{e\}$.

Ex

Let $S$ be a set.

Then

(1) The power set $P\,(S)$ of $S$ is a semigroup
with union(intersection) operation.

(2) $G = \{f | f : S \to S\}$ is a semigroup under composition operation.

(3) $G$ is the set of strings of finite length is a semigroup
with concatenation operation.

Ex

$G$ is the set of $n \times n$ matrices over $\mathbb{R}$ with product operation is a semigroup.

Definition

(1) A group is a monoid $G$ such that for all $a \in G$,

there exists $a^{-1} \in G$ such that $aa^{-1} = e = a^{-1}a$.

(2) A group $G$ is abelian if $ab = ba$ for $a, b \in G$.

Ex

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are groups under $+$ and $e = 0$.

Ex

Let $S$ be a set, $G = \{f | f : S \to S \text{ is } 1-1 \text{ and onto}\}$,

is a group under composition.

If $|S| = n$, write $S_n$ for $G$.

If $S = \{1, 2, \cdots, n\}$, we write $f = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ f_1 & f_2 & f_3 & \cdots & f_n \end{pmatrix}$.

e.g. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}$.

Note

(1) $|S_n| = n!$.

(2) $e$ is unique in the definition of monoid.

(3) $a^{-1}$ is unique for each $a \in G$ in the definition of group.

Ex

$\mathbb{Z}_n = \{0, 1, 2, ..., n-1\}$.

$\mathbb{Z}_n$ is a group under the additor $+$ in $\mathbb{Z}$ modulo $n$.