Def : Suppose $H<G$. $G/H := \{Hg \mid g \in G\}$ **(a set of sets)** is the set of right cosets of H in G.

Theorem : Suppose $H<G$. Then

(1) Hh = H for $h \in H$

(2) If $|H| < \infty$ then $|Hg| = |H|$ for any $g \in G$.

(3) $Hg = Hg' \Leftrightarrow g'g^{-1} \in H$ for any $g, g' \in G$

(4) $Hg = Hg'$ or $Hg \cap Hg' = \phi$

Pf : (1) ($\subseteq$) Clear.

($\supseteq$) Pick $a \in H$. Then $ah^{-1} \in H$. Hence $a \in Hh$.

(2) Define $f : Hg \to H$ by $f(a \cdot g) = a$ for $a \in H$. Check f is well-defined, 1-1, onto.

(3) ($\Rightarrow$) Clear.

($\Leftarrow$) Suppose $g'g^{-1} \in H$, say $g'g^{-1} = h \in H$

Then $Hg = Hhg = Hg'$.

(4) Suppose $Hg \neq Hg'$ and $Hg \cap Hg' \neq \phi$.

Then $g'g^{-1} \notin H$ by (3) and $hg = h'g'$ for some $h, h' \in H$

Hence $g'g^{-1} = (h')^{-1}h \in H$.

Recall : $H<G$, then

(1) Ha is called a right coset of H in G , for $a \in G$.

(2) $|H| = |Ha|$ if $|G| < \infty$.

(3) $Ha = Hb$ or $Ha \cap Hb = \phi$ for $a, b \in G$

Def : For $H<G$, let $[G:H]$ denote the of right cosets of H in G. $[G:H]$ is called the index of H in G.

Cor : (Lagrange Theorem) $|G| < \infty \Rightarrow [G:H] = {}^{|G|}\!\big/_{|H|}$. In particular, $|H| \,\big|\, |G|$.

Theorem : H, K < G and $|G| < \infty \Rightarrow |HK| = |H||K| \big/ |H \cap K|$

Pf: $K = (H \cap K)t_1 \cup (H \cap K)t_2 \cup \cdots \cup (H \cap K)t_s$ ($\cup$ means disjoint union)

for some $t_i \in K$.

Then $HK = H(H \cap K)t_1 \bigcup H(H \cap K)t_2 \bigcup \cdots \bigcup H(H \cap K)t_s$

$= Ht_1 \cup Ht_2 \cup \cdots \cup Ht_s$.

Note $Ht_i \cap Ht_j = \phi$ or $Ht_i = Ht_j$.

Suppose $Ht_i = Ht_j$.

Then $t_j t_i^{-1} \in H \cap K$. Thus $(H \cap K)t_j = (H \cap K)t_i$ i.e. $i = j$

Hence $|HK| = |H| \cdot s = |H| \cdot |K| \big/ |H \cap K|$

Note : $HK \subseteq \langle H, K \rangle < G$