1.5 Normality,Quotient Groups and Homomorphisms

Theorem:
suppose $N < G$
TFAE
(1)$gN = Ng$ for all $g \in G$
(2)$g^{-1}Ng = N$ for all $g \in G$
(3)$g^{-1}Ng$ for all $g \subseteq G$
If (1)⇒(3)holds ,we say N is normal in G and denoted by $N \lhd G$
pf:
(2)⇒(3)clear
(3)⇒(1)$gN = gNg^{-1} \subseteq Ng$ similarly $Ng \subseteq gN$
(1)⇒(2)$g^{-1}Ng = g^{-1}gN = N$ Theorem:
$H < G$,$N \lhd G \Rightarrow HN = NH < G$
PF:
(1)$hn = hnh^{-1} \in NH$ for $h \in H$ and $n \in H$. This proves $HN \subseteq NH$, similarly $NH \subseteq HN$
(2)pick $hn, h^{'}n^{'} \in HN$. Then $(hn)(h^{'}n^{'})^{-1} = hnn^{'-1}h^{'-1} = hn^{'-1}n^{'}nh^{'-1}h^{'}n^{'-1}h^{'-1} \in HN$. Hence $HN < G$
Theorem:
$N \lhd G$. Define a product $\circ$ on G/N by $Na \circ Nb = Nab$ for $a, b \in G$.Then G/N is a group under this product and $e = N$. G/N is called the quotient of N by G.
pf:
We need to show the product is well-defined. Suppose $Na = Na^{'}$ and $Nb = Nb^{'}$ for $a, b, a^{'}, b^{'} \in G$.
(claim: $Nab = Na^{'}b^{'}$)
Then $a^{'}a^{-1} \in N$ and $b^{'}b^{-1} \in N$. So, $a^{'}a^{-1}ab^{'}b^{-1}a^{-1} \in N$. Thus $a^{'}b^{'}b^{-1}a^{-1} = a^{'}b^{'}(ab)^{-1} \in N$.Hence $Nab = Na^{'}b^{'}$. The remaining is clear.
Theorem:(First Fundamented Tehorem of Homomorphism)
Let $f : G \to H$ be a homomorphism of groups then $kerf \lhd G$ and $f(G)$ is isomorphic to $G/kerf$.
pf:
(1)$f(g^{-1}(kerf)g) = f(g)^{-1}\{e\}f(g) = \{e\}$. i.e. $g^{-1}(kerf)g \subseteq kerf$ for $g \in G$. This proves $kerf \lhd G$.
(2)Define a map $\tilde{f} : G/kerf \to f(G)$ by $\tilde{f}((kerf)g) = f(g)$ for $g \in G$. It is routine to check $\tilde{f}$ is well-defined, homomorphism, 1-1, and onto.