1.6 Symmetric, Alternating and Dihedreal Groups

For $\{i_1, i_2, \cdots, i_r\} \subseteq \{1, 2, \cdots, n\}$, we use $(i_1, i_2, \cdots, i_r)$ to denote the permutation $\begin{pmatrix} 1 & 2 & \cdots & i_j & \cdots & a & \cdots & n \\ 1 & 2 & \cdots & i_{j+1} & \cdots & a & \cdots & n \end{pmatrix}$, where $a \notin \{i_1, i_2, \cdots, i_r\}$ and $j+1$ is modulo r.

Example 1. *(1,2,3,4) represents* $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & n \\ 2 & 3 & 4 & 1 & 5 & 6 & \cdots & n \end{pmatrix}$.

Example 2.

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} \\ &= (1,2)(3,4) \quad \textit{disjoint cycles} \\ &= (3,4)(1,2) \end{aligned}$$

$(i_1, i_2, \cdots, i_r)$ is called a cycle of length r. If r=2, it is called a transposition.

Note: Disjoint cycles commute.

Theorem 1. *Every permutation in S_n can be written as a product of disjoint cycles uniquely up to the order of cycles.*

Example 3.

$$\begin{aligned} & \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & 4 & 6 & 7 & 8 & 5 & 9 \end{pmatrix} \\ =\ & (1,3,2)(4)(5,6,7,8)(9) \\ =\ & (1,3,2)(5,6,7,8) \end{aligned}$$

Proof. For each $\sigma \in S_n$, σ gives a directed graph on $\{1, 2, \cdots, n\}$ with arc $i \to \sigma(i)$. This digraph has "indegree" 1 since σ is $1-1$ and $n < \infty$. It has "outdegree" 1 since σ is a function. Hence, the directed graph is a disjoint union of directed cycles. □

Definition 1. *S_n is called the symmetric group on $\{1, 2, \cdots, n\}$.*

Example 4.

$$\begin{aligned} & (123456) \\ = \; & (16)(15)(14)(13)(12) \\ = \; & (12)(23)(34)(56)(56) \end{aligned}$$

Theorem 2. *No permutation $\sigma \in S_n$ can be written as a product of odd number of transpositions and as a product of even number of transpositions.*

Proof. Consider $f(x_1, \cdots, x_n) := \prod_{1 \le i < j \le n} (x_i - x_j)$.

For $\sigma \in S_n$, define

$$\begin{aligned} \sigma f(x_1, x_2, \cdots, x_n) &= f(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)}) \\ &= \prod_{1 \le i < j \le n} (x_{\sigma(i)} - x_{\sigma(j)}) \end{aligned}$$

Observe
(0) $\sigma f = \pm f$ for $\sigma \in S_n$.
(1) $(\sigma\tau) f = \sigma(\tau f)$ for $\sigma, \tau \in S_n$.
(2) $(a_1, a_2) f = -f$ for $(a_1, a_2) \in S_n$.
(Taking all possible situations for i, j given a_1 and a_2. The detail can be found in the textbook.)
(3) $(a_1, b_1)(a_2, b_2) \cdots (a_t, b_t) f = (-1)^t f$.
The theorem follows form (3). □