Def: Let A_n be the set of even permutations in S_n.

Theorem:(1)$A_n \lhd S_n$(2)$[S_n : A_n] = 2$ if $n \geq 2$

Prove:

(1)$\sigma^{-1}\tau\epsilon A_n$ for all $\sigma, \tau\epsilon A_n \Longrightarrow A_n \leq S_n$ $\sigma^{-1}\tau\sigma\epsilon A_n$ if $\sigma\epsilon S_n$ and $\tau\epsilon A_n \Longrightarrow A_n \lhd S_n$.

(2)Define $f : S_n \longrightarrow \mathbb{Z}_2$ by

$$f(\sigma) = \begin{cases} 0, & \text{if } \sigma \text{ is even} \\ 1, & \text{if } \sigma \text{ is odd} \end{cases}$$

Then f is an epimorphism with kernel A_n. By first homomorphism, $S_n \diagup A_n \cong \mathbb{Z}_2$, hence $[S_n : A_n] = 2$. $\Box$

Def: G is simple if g has no normal subgroups except $\{e\}$ and G.

Cor: S_n is not simple for $n \geq 2$.

Note: $g(1, 2, ..., t)g^{-1} = (g(1), g(2), ..., g(t))$ for $g\epsilon S_n$.

ex: $K_4 = \{e, (12)(34), (14)(23), (13)(24)\} < A_4$.

$g(12)(34)g^{-1} = g(12)g^{-1}g(34)g^{-1} = (g(1)g(2))(g(3)g(4))\epsilon K_4$. Hence $K_4 \lhd A_4$.

Fact: A_n is simple if $n \neq 4$.

Def: Fix $n\epsilon\mathbb{N}$, $\sigma = (1, 2, 3, ..., n), \tau = (2, n)(3, n-1)(4, n-2)...$,

$D_n =< \sigma, \tau >$ is called the Dihedral group of rank n.

Notation: $|\sigma| = | < \sigma > |$ for $\sigma\epsilon S_n$.

Property:

(1)$|\sigma| = n$

(2)$|\tau| = 2$

(3)$\sigma\tau = \tau\sigma^{-1}$

(4)$D_n = 2n$

Prove:

(1)(2) are clear.

(3):

$\sigma\tau(1) = 2 = \tau\sigma^{-1}(1)$, $\sigma\tau(2) = 1 = \tau\sigma^{-1}(2)$, $\sigma\tau(n) = 3 = \tau\sigma^{-1}(n)$.

For $3 \leq i \leq n-1$, $\sigma\tau(i) = \sigma(n+2-i) = n+3-i$ and $\tau\sigma^{-1}(i) = 2(i-1) = n+2-(i-1) = n+3-i$. Hence $\sigma\tau = \tau\sigma^{-1}$. (4):
$D_n = \{\sigma^i\tau^j | 0 \leq i \leq n-1, 0 \leq j \leq 1\}$ by (3).
If $\sigma^i\tau^j = \sigma^k\tau^l$ then $\sigma^{i-k} = \tau^{l-j} = e$ for $0 \leq i, k \leq n-1$, $0 \leq l, j \leq 1$. Hence $i = k, l = j$.Thus $|D_n| = 2n$. $\square$