1.8 Direct product and direct sum

Definition. *Let G_1 and G_2 be groups. $G_1 \times G_2$ is called the direct product of G_1 and G_2.*
If G_1 and G_2 are abelian, we called a direct product as a direct sum.

Note:
1. $\{e\} \times G_2, G_1 \times \{e\} \triangleleft G_1 \times G_2$,
2. $\Pi_i : G_1 \times G_2 \to G_i$ defined by $\Pi_i(g_1, g_2) = g_i$ for $i = 1, 2$
3. $i_j : G_j \to G_1 \times G_2$ defined by

$$i_j(g) = \begin{cases} (g, e) & \text{if } j = 1, \\ (e, g) & \text{if } j = 2 \end{cases}$$

is called the j^{th} inclusion map of $G_1 \times G_2$
4. Π_1, Π_2, i_1, i_2 are group homomorphisms.

Theorem. *Let G_1 and G_2 be groups, then there exists a unique group, up to isomorphism, with homomorphisms $\phi_i : G \to G_i$ satisfying the following "product rule": For any group G' and any homomorphisms, $\phi'_i : G' \to G_i$,there exists a unique homomorphism $\phi : G' \to G$ s.t. $\phi_i \phi = \phi'_i$ for i=1,2. In fact $G \cong G_1 \times G_2$.*

$$\begin{array}{ccccc} & & G & & \\ & \phi_1 \swarrow & & \searrow \phi_2 & \\ G_1 & & \phi \uparrow & & G_2 \\ & \phi'_1 \nwarrow & & \nearrow \phi'_2 & \\ & & G' & & \end{array}$$

Note:
$G_1 \times G_2$ is the "essetial part" if considering homomorphisms from any groups to G_1 and to G_2.

Proof. (Existence) Choose $G = G_1 \times G_2$ and $\phi_1 = \Pi_1$, $\phi_2 = \Pi_2$. For any G', ϕ'_1, ϕ'_2 as described, define $\phi : G' \to G = G_1 \times G_2$ by $\phi(g') = (\phi'_1(g'), \phi'_2(g'))$. Then clearly, $\Pi_i \phi(g') = \phi'_i(g')$ and no other way to define ϕ with this property.

(Uniqueness) Suppose (G, ϕ_1, ϕ_2) and (G', ϕ'_1, ϕ'_2) are two such pairs. Then there exists a homomorphism $\phi : G' \to G$ and a homomorphism $\phi' : G \to G'$ such that $\phi'_i = \phi_i \phi$ and $\phi_i = \phi'_i \phi'$. Then $\phi_i \phi \phi' = \phi'_i \phi' = \phi_i$ for i=1,2. Let $I_G : G \to G$ be the identity, clear $\phi_i I_G = \phi_i$. By the uniqueness of I_G, we have $\phi \phi' = I_G$, then $\phi : G \to G'$ is an isomorphism. □