## 1.8 Direct product and direct sum

**Definition.** Let  $G_1$  and  $G_2$  be groups.  $G_1 \times G_2$  is called the direct product of  $G_1$  and  $G_2$ .

If  $G_1$  and  $G_2$  are abelian, we called a direct product as a direct sum.

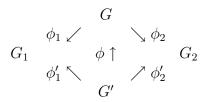
Note:

1.  $\{e\} \times G_2, G_1 \times \{e\} \triangleleft G_1 \times G_2,$ 2.  $\Pi_i : G_1 \times G_2 \rightarrow G_i$  defined by  $\Pi_i(g_1, g_2) = g_i$  for i = 1, 23.  $i_j : G_j \rightarrow G_1 \times G_2$  defined by

$$i_j(g) = \begin{cases} (g, e) & \text{if } j = 1, \\ (e, g) & \text{if } j = 2 \end{cases}$$

is called the  $j^{th}$  inclusion map of  $G_1 \times G_2$ 4.  $\Pi_1, \Pi_2, i_1, i_2$  are group homomorphisms.

**Theorem.** Let  $G_1$  and  $G_2$  be groups, then there exists a unique group, up to isomorphism, with homomorphisms  $\phi_i : G \to G_i$  satisfying the following "product rule": For any group G' and any homomorphisms,  $\phi'_i : G' \to G_i$ , there exists a unique homomorphism  $\phi : G' \to G$  s.t.  $\phi_i \phi = \phi'_i$  for i=1,2. In fact  $G \cong G_1 \times G_2$ .



Note:

 $G_1 \times G_2$  is the "essetial part" if considering homomorphisms from any groups to  $G_1$  and to  $G_2$ .

*Proof.* (Existence) Choose  $G = G_1 \times G_2$  and  $\phi_1 = \Pi_1$ ,  $\phi_2 = \Pi_2$ . For any G',  $\phi'_1$ ,  $\phi'_2$  as described, define  $\phi : G' \to G = G_1 \times G_2$  by  $\phi(g') = (\phi'_1(g'), \phi'_2(g'))$ . Then clearly,  $\Pi_i \phi(g') = \phi'_i(g')$  and no other way to define  $\phi$  with this property. (Uniqueness) Suppose  $(G, \phi_1, \phi_2)$  and  $(G', \phi'_1, \phi'_2)$  are two such pairs. Then there exists a homomorphism  $\phi: G' \to G$  and a homomorphism  $\phi': G \to G'$ such that  $\phi'_i = \phi_i \phi$  and  $\phi_i = \phi'_i \phi'$ . Then  $\phi_i \phi \phi' = \phi'_i \phi' = \phi_i$  for i=1,2. Let  $I_G: G \to G$  be the identity, clear  $\phi_i I_G = \phi_i$ . By the uniqueness of  $I_G$ , we have  $\phi \phi' = I_G$ , then  $\phi: G \to G'$  is an isomorphism.  $\Box$