Example:$X = \{a, b\}$, $Y = \{aba^{-1}b^{-1}\}$ implice that $F(x)/N_y \cong (\mathbb{Z} \times \mathbb{Z}, +)$
proof: Let ϕ:$F(x) \to \mathbb{Z} \times \mathbb{Z}$ be a map defined by ϕ(a) = (1,0) , ϕ(b) = (0,1)
Note:$\phi(a^{i_1}b^{j_1}a^{i_2}b^{j_2}\cdots a^{i_t}b^{j_t}) = (\sum_{k=1}^{t} i_k \ , \ \sum_{k=1}^{t} j_k)$
Then ϕ is epirmorphism,
By first homomorphism theorem,$F(x)/N_y \cong (\mathbb{Z} \times \mathbb{Z})$
We need to check that $ker\phi = N_y$.
$\phi(aba^{-1}b^{-1}) = (0,0) \ \Rightarrow N_y \subseteq ker\phi$

Now, we claim: $a^n b^m a^{-n} b^{-m} \in N_y$

$aba^{-1}b^{-1} \in Y \subseteq N_y$
$\Rightarrow b^{-1}aba^{-1}b^{-1}b \in N_y$
$\Rightarrow a^{-1}b^{-1}ab \in N_y$
$\Rightarrow a^{-1}bab^{-1} \ = \ (ba^{-1}b^{-1}a)^{-1} \in N_y$
$\Rightarrow (b^{-1}aba^{-1})^{-1} \ = \ ab^{-1}a^{-1}b \in N_y$

By eaplacing a by a^{-1}, b by b^{-1}, if necessary, we assume $m, n\, 0$.
Then $a^n b^m a^{-n} b^{-m} \ = \ aa^{n-1}b^m a^{-(n-1)}b^{-m}bb^{m-1}a^{-1}b^{-(m-1)}aa^{-1}b^{-1}abb^{-1}a^{-1} \in N_y$

Next, we claim: $ker\phi \ \subseteq \ N_y$

Suppose not, pick $a^{i_1}b^{j_1}a^{i_2}b^{j_2}\cdots \ \in \ ker\phi - N_y$,
then $b^{j_1}a^{i_1}b^{-j_1}a^{-i_1}a^{i_1}b^{j_1}a^{i_2}b^{j_2}\cdots \ \in \ ker\phi - N_y$
Hence, $b^{j_1}a^{i_1}a^{i_2}b^{j_2}\cdots \ \in \ ker\phi - N_y$
keep doing this, we will obtained $e \in \ ker\phi - N_y$.
by using $\sum i_k = 0 = \sum j_k$,a contradiction.
So, $ker\phi \ = \ N_y$

Fact:
(1)Let G_1 and G_2 be groups. Then "the free group generated by G_1 and G_2 subject to origional properties in G_1 and G_2" is the unique group, up to isomorphism, that satisfies the "coproduct" rule of G_1 and G_2.
(2)Any subgroup of a free group is isomorphic to another free group with possible more generators.

Example:$X = \{a, b\}$,$N = \{a^{i_1}b^{j_1}a^{i_2}b^{j_2}\cdots \ \in \ F(X) : \sum i_k + \sum j_k$ is even.$\}$,then $N < F(X)$
proof:Note:$N = < a^2, b^2, ab >$
Let $X' = \{x, y, z\}$ and $\phi : F(X') \to N$, defined by $\phi(x) = a^2, \phi(y) = b^2, \phi(z) = ab$.
It's routine to check that ϕ is an isomorohism.