Advanced Algebra II Class Note

2.7 Nilpotent And Solvable Groups (2)

05/21

Lemma

$H < G$ and G is solvable $\Rightarrow$ H is solvable.

Proof

$H^{(n)} \subset G^{(n)} = \langle e \rangle$ for some n.

Theorem

Let $\phi : G \to H$ be an epihomomorphism and G is solvable.

Then H is solvable.

Proof

Suppose $G^{(n)} = \langle e \rangle$.

Then $H^{(n)} = \phi(G)^{(n)} = \phi\left(G^{(n)}\right) = \phi(\langle e \rangle) = \langle e \rangle$.

Theorem

Suppose $N \lhd G$ and G is solvable.

Then G/N is solvable.

Proof

$\bar{g}\bar{h}\bar{g}^{-1}\bar{h}^{-1} = \overline{ghg^{-1}h^{-1}} \in G'N/N$ for $\bar{g}, \bar{h} \in G/N$.

Then $(G/N)' \subseteq G'N/N$.

Hence $(G/N)^{(n)} \subseteq G^{(n)}N/N = \langle \bar{e} \rangle$ for some n.

Theorem

Suppose $N \lhd G$ and N, G/N are solvable.

Then G is solvable.

Proof

Choose n such that $(G/N)^{(n)} = \langle \bar{e} \rangle$.

Then $G^{(n)} \subseteq N$.

Choose m such that $N^{(m)} = \langle e \rangle$.

Then $G^{(n+m)} = \langle e \rangle$.

Corollary

If $n \geq 5$, then S_n is not solvable.

Proof

If S_n is solvable, then A_n is solvable.

Note $A_n' \lhd A_n$, A_n is simple and not abelian.

Hence $A_n' = A_n$.

Then $A_n^{(m)} = A_n \neq \langle e \rangle$ for all n.

Thus A_n is not solvable, a contradiction.