2.7 Nilpotent and Solvable Subgroups

Definition. $G'(:= [G,G]) =< ghg^{-1}h^{-1}|g,h \in G >$ is called the commutation subgroup of G.

Example. G is abelian $.\Rightarrow G' =< e >$

Lemma. $G' \lhd G$.

Proof. *We need to prove $ghg^{-1} \in G'$ for any $h \in G', g \in G$.*
It sufficiences to assume $h = g_1 h_1' g_1^{-1} h_1^{-1}$ for $g_1, h_1 \in G$.
Then $gg_1h_1g_1^{-1}h_1^{-1}g^{-1}$
$= gg_1g^{-1} \quad gh_1g^{-1} \quad gg_1^{-1}g^{-1} \quad gh_1^{-1}g^{-1}$
$= (gg_1g^{-1})(gh_1g^{-1})(gg_1g^{-1})^{-1}(gh_1g^{-1})^{-1} \in G.\square$

Lemma. *G/G' is abelian.*

Proof. $\bar{g}\bar{h}\bar{g}^{-1}\bar{h}^{-1} = \overline{ghg^{-1}h^{-1}} = \bar{e}$ *for any $g,h \in G.\square$*

Proposition. *$N \lhd G$. Then G/N is abelian $\Leftrightarrow G' < N$.*

Proof. $(\Rightarrow)$ $\overline{ghg^{-1}h^{-1}} = \bar{g}\bar{h}\bar{g}^{-1}\bar{h}^{-1} = \bar{e} = eN \in G/N$ *for $g,h \in G$*
Hence $ghg^{-1}h^{-1} \in N$ for any $g,h \in G$

$(\Leftarrow)$ *$(G/G')/(N/G') \approxeq G/N$ is abelian since G/G' is abelian.* $\square$

Definition. $G^{(1)} = G'$
$G^{(n+1)} = [G^{(n)}, G^{(n)}]$
$G^{(n)}$ is called the n^{th} derived series.

Definition. G is solvable if $G^{(n)} =< e >$ for some $n \in \mathbb{N}$.

Definition. Let $Z(G)$ be the center of G.
Define $Z_1(G) = Z(G)$.
$Z_{n+1}(G)$ is the preimage of $Z(G/Z_n(G))$
i.e. $Z_{n+1}(G)/Z_n(G) = Z(G/Z_n(G))$.

Example. G is abelian. $\Rightarrow G = Z_1(G) = Z_2(G) = \cdots$

Note. $Z_1(G) \lhd Z_2(G) \lhd Z_3(G) \lhd \cdots$

Definition. G is nilpotent if $Z_n(G) = G$ for some $n \in \mathbb{N}$.

Theorem. *G is nilpotent* $\Rightarrow$ *G is solvable.*

Proof. *Suppose* $< e >= Z_0(G) \lhd Z_1(G) \lhd Z_2(G) \lhd Z_3(G) \lhd \cdots \lhd Z_{n-2}(G) \lhd Z_{n-1}(G) \lhd Z_n(G) = G$.
Then $Z_{n-i}(G) \geqslant (Z_{n-i+1}(G))' \geqslant (Z_{n-i+2}(G))' \geqslant G^{(i)}$, *since* $Z_{n-i+1}(G)/Z_{n-i}(G)$ *is abelian.*
Hence $< e >= Z_0(G) \geqslant G^{(n)}$.□