Advanced Algebra II Class Note

2.3 The Krull-Schmidt Theorem

03/26

Definition

$G = G_1 \times G_2 \times \cdots \times G_t$

if (1) $G_i \lhd G$

(2) $G = G_1 G_2 \cdots G_t$

(3) $G_{\sigma(1)} \cap G_{\sigma(2)} G_{\sigma(3)} \cdots G_{\sigma(t)} = \{e\}$

for $\sigma \in S_t$.

In this case, we say G is the (inner) direct product of G_1, G_2, ..., G_t.

Ex

$$\begin{aligned}
\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 &:= \{(a,b,c) \mid a,b,c \in \mathbb{Z}_2\} \\
&= \langle(1,0,0)\rangle \times \langle(0,1,0)\rangle \times \langle(0,0,1)\rangle \\
&= \langle(1,0,0)\rangle \times \langle(1,1,1)\rangle \times \langle(0,0,1)\rangle \\
&\neq \langle(1,0,0)\rangle \times \langle(0,1,0)\rangle \times \langle(0,0,1)\rangle \times \langle(1,1,1)\rangle.
\end{aligned}$$

Note

(1) In previous example, $\langle(1,1,1)\rangle \cap \langle(1,0,0)\rangle = \{(0,0,0)\}$.

Hence the assumption (3) can not be changed to $G_i \cap G_j = \{e\}$ for $i \neq j$.

(2) If $a \in G_i$, $b \in G_j$,

Since $ba^{-1}b^{-1} \in G_i$ and $aba^{-1} \in G_j$,

$$aba^{-1}b^{-1} \in G_i \cap G_j \subseteq \{e\} \text{ if } i \neq j.$$

(3) $G = G_1 \times G_2 \times \cdots \times G_t$

$\Leftrightarrow G = G_{\sigma(1)} \times G_{\sigma(2)} \times \cdots \times G_{\sigma(t)}$ for any $\sigma \in S_t$.

(4) $G = G_1 \times G_2 \times G_3$

$= (G_1 \times G_2) \times G_3$

$= G_1 \times (G_2 \times G_3).$

(5) $G = G_1 \times G_2 \times \cdots \times G_t$

$\Rightarrow G \cong G_1 \times G_2 \times \cdots \times G_t.$

Definition

G is indecomposable

if $G \neq H \times K$ for any nontrivial normal subgroups $H, K < G$.

Ex

$\mathbb{Z}$, $\mathbb{Z}_{p^n}$, S_3 are indecomposable, where p is a prome.

Ex

If $(m, n) = 1$,

then $\mathbb{Z}_{m \times n} = \{0, 1, 2, \ldots, mn - 1\}$

$= \langle m \rangle \times \langle n \rangle.$

(Hint: $am + bn = 1$ for some $a, b \in \mathbb{Z}$.)

Definition

We say G satisfies ascending chain condition on normal subgroups (ACCN)

if for any $G_i \triangleleft G$ with $G_1 < G_2 < G_3 < \cdots$,

there exists $n \in \mathbb{N}$, s.t. for all $i \geq n$, $G_i = G_n$.

Definition

We say G satisfies descending chain condition on normal subgroups (DCCN)

if for any $G_i \triangleleft G$ with $G_1 > G_2 > G_3 > \cdots$,

there exists $n \in \mathbb{N}$, s.t. for all $i \geq n$, $G_i = G_n$.

Ex

$|G| < \infty \Rightarrow G$ satisfy both ACCN and DCCN.

Ex

$\mathbb{Z}$ has ACCN, but does not have DCCN.

(Hint: Each subgroup of $\mathbb{Z}$ has the form $\langle m \rangle$ for $m \in \mathbb{Z}$.)

Theorem

G satisfies ACCN or DCCN

$\Rightarrow G$ is the direct product of finite indecomposable normal subgroups.

Proof

If G is indecomposable, we are done, since $G = G$.

Suppose $G = G_1 \times H_1$, where $\{e\} \neq G_1,\ H_1 \lhd G$.

At least one of G_1 and H_1 is not a direct product

of finite indecomposable normal subgroups, say G_1,

otherwise we are done.

Let G_1 be the role of G to have $G_1 = G_2 \times H_2$ keep doing in this way,

we have
$$
\begin{aligned}
G &= G_1 \times H_1 \\
&= G_2 \times H_2 \times H_1 \\
&= G_3 \times H_3 \times H_2 \times H_1 \\
&\vdots \\
&= G_n \times H_n \times H_{n-1} \times \cdots \times H_1.
\end{aligned}
$$

Hence $H_1 \lneqq H_1 \times H_2 \lneqq H_1 \times H_2 \times H_3 \lneqq \cdots$, a contradiction to ACCN,

and $G_1 \gneqq G_2 \gneqq G_3 \gneqq \cdots$, a contradiction to DCCN.