2.3 The Krull-Schmidt Theorem

Definition 3.1. An *endomorphism* of G is a homomorphism from G into G.

Note 3.2. An automorphism is an endomorphism + 1-1 + onto.

Lemma 3.3. *Suppose that G has* ACCN *property and $f \in \text{End}(G)$. T.F.A.E.*

1. $f \in \text{Aut}(G)$
2. *f is an epimorphism*

Proof. ($1 \Rightarrow 2$) is trivial.

($2 \Rightarrow 1$, already know onto, need to show 1-1) Note $\ker f^n \subseteq \ker f^{n+1}$ for $n \in \mathbb{N}$. By ACCN, there exists $m \in \mathbb{N}$ such that $\ker f^i = \ker f^m$ for all $i \geq m$. Pick $a \in \ker f$. By assumption 2, there exists $b \in G$ such that $f^m(b) = a$. ($\because f$ is onto, $\therefore f^m$ is onto) Then $e = f(a) = f^{m+1}(b)$. Hence $b \in \ker f^{m+1} = \ker f^m$. Thus $a = f^m(b) = e$. We proved $\ker f = \{e\}$. Then f is 1-1, and hence an automorphism. □

Lemma 3.4. *(dual) Suppose that G has DCCN property and $f \in \text{End}(G)$. T.F.A.E.*

1. $f \in \text{Aut}(G)$
2. *f is an monomorphism*

Proof. ($1 \Rightarrow 2$) is trivial.

($2 \Rightarrow 1$, already know 1-1, need to show onto) Note $\text{Img} f^n \supseteq \text{Img} f^{n+1}$ for $n \in \mathbb{N}$. By DCCN, there exists $m \in \mathbb{N}$ such that $\text{Img} f^i = \text{Img} f^m$ for all $i \geq m$. Pick $a \in G$. Then there exists $b \in G$ such that $f^{m+1}(b) = f^m(a)$. By assumption 2, $f^m(a) = f^{m+1}(b) = f^m(f(b))$ implies $a = f(b)$. ($\because f$ is 1-1, $\therefore f^m$ is 1-1) Then f is onto, and hence an automorphism. □

Definition 3.5. $f \in End(G)$ is *normal* if $a^{-1}f(b)a = f(a^{-1}ba)$ for any $a, b \in G$.

Example 3.6. $\pi_1 : G_x \times G_2 \to G_1 \times G_2$, and $\pi_1((a, b)) = (a, e)$ is normal.

Proof. $(h_1, h_2)^{-1}\pi_1(a, b)(h_1, h_2) = (h_1, h_2)^{-1}(a, e)(h_1, h_2) = (h_1^{-1}ah_1, h_2^{-1}eh_2)$
$= \pi_1((h_1, h_2)^{-1}(a, b)(h_1, h_2))$ □

Note 3.7. $f \in \text{End}(G)$ *is normal* $\Rightarrow \text{Img} f^n \lhd G$ *for all* $n \in \mathbb{N}$. $(a^{-1}f^2(b)a = a^{-1}f(f(b))a = f(a^{-1}f(b)a) = f(f(a^{-1}ba)) = f^2(a^{-1}ba))$

Lemma 3.8. *Suppose that G satisfies both ACCN and DCCN, and $f \in \text{End}(G)$ is normal. Then $G = \ker f^n \times \text{Img} f^n$ for some $n \in \mathbb{N}$.*

Proof. Pick $n \in \mathbb{N}$ such that for all $i \geq n$ we have $\ker f^i = \ker f^n$ and $\text{Img} f^i = \text{Img} f^n$.

Claim: $G = \ker f^n \times \text{Img} f^n$

1. Pick $a \in \ker f^n \cap \text{Img} f^n$. Then $f^n(a) = e$ and $f^n(b) = a$ for some $b \in G$. Hence $f^{2n}(b) = f^n(a) = e$. Thus $b \in \ker f^{2n} = \ker f^n$. Then $a = f^n(b) = e$.
2. Pick any $g \in G$. Pick $h \in G$ such that $f^{2n}(h) = f^n(g)$. Then $g = gf^n(h^{-1})f^n(h)$. Since $f^n(h) \in \text{Img} f^n$. We want to show $gf^n(h^{-1}) \in \ker f^n$. Since $f^n(gf^n(h^{-1})) = f^n(g) \cdot f^{2n}(h^{-1}) = f^n(g) \cdot f^n(g^{-1}) = e$, we are done.

□

Definition 3.9. $f \in \operatorname{End}(G)$ is *nilpotent* if there exists $n \in \mathbb{N}$ such that $f^n(a) = e$ for all $a \in G$.

Example 3.10. $G = \mathbb{Z}_8$. $f(a) = a + a$ for $a \in \mathbb{Z}_8$. $\Rightarrow f^3(a) = 0$ for all $a \in \mathbb{Z}_8$.

Corollary 3.11. *Let G be indecomposable, ACCN, DCCN, and $f \in \operatorname{End}(G)$ is normal. Then f is nilpotent or f is an automorphism.*

Proof. Pick $n \in \mathbb{N}$ such that $G = \ker f^n \times \operatorname{Img} f^n$. Since G is indecomposable, either $\ker f^n = G$ or $\operatorname{Img} f^n = G$. In the first case, f is nilpotent, and the latter is onto (then automorphism by previous lemma). □