Definition. Let $f, g : G \mapsto G$ be function (not necessary endomorphism). Define a function $f + g : G \mapsto G$ by $(f + g)(a) = f(a) \cdot g(a)$.

Note. *1. $f + g \neq g + f$ if g is not abelian.*

2. $f, g \in End(G) \not\Rightarrow f + g \in End(G)$

3. $G^G := \{f | f : G \mapsto G$ is a function $\}$ is a group under $+$.

4. $O_G : G \mapsto G$ is defined by $O_G(a) = e$ for $a \in G$

5. $-f : G \mapsto G$ is defined by $(-f)(a) = f(a^{-1})$

6. For $f, g, h \in G^G$. $(f + g)h = f \cdot h + g \cdot h$

Proof. *6.*

$$\begin{aligned} (f+g)\cdot h(a) &= (f+g)(h(a)) \\ &= f(h(a)) \cdot g(h(a)) \\ &= (f \cdot h)(a) \cdot (g \cdot h)(a) \\ &= (f + h \cdot g + h)(a) \end{aligned}$$

Example. $G = S_3$
$f(\sigma) = (123)^{-1}\sigma(123)$
$g(\sigma) = (12)^{-1}\sigma(12)$
$\Rightarrow 1. f, g \in End(G)$
$2. (f + g)(\sigma) = (123)\sigma(123)^{-1}(132)\sigma(132)^{-1} = (123)\sigma(123)\sigma(123)$

Use $f + g(123) \neq f + g(12)(f + g)(13)$ to check $f + g \notin End(G)$

Example. $\prod_i : G_1 \times G_2 \times G_3 \to G_1 \times G_2 \times G_3$
$\prod_1 : (a, b, c) \to (a, e, e)$
$\prod_2 : (a, b, c) \to (e, b, e)$
$\prod_i \in End(G_1 \times G_2 \times G_3)$
$\Rightarrow 1. \prod_1 + \prod_2(a, b, c) = (a, b, e)$
$2. \prod_1 + \prod_2 \in End(G_1 \times G_2 \times G_3))$

Remark. G has $ACCN, DCCN$ and is indecomposable.$f \in End(G)$ is normal.
$\Rightarrow f$ is nilpotent or f is an automorphism.

Lemma. *Suppose $f, g, f + g \in End(G)$ and f, g are normal. Then $f + g$ is normal.*

Proof. $a^{-1}(f + g(b))a$
$= a^{-1}f(b)g(b)a$
$= a^{-1}f(b)aa^{-1}g(b)a$
$= f(a^{-1}ba)g(a^{-1}ba)$
$= (f + g)(a^{-1}ba).\square$

Lemma. *Let G be indecomposable satisfying $ACCN, DCCN$. Suppose $f, g, f+g \in End(G)$,and suppose f, g are nilpotent. Then $f+g$ is nilpotent.*

Proof. *Suppose that $f+g$ is not nilpotent.*
Then $f+g \in Aut(G)$.
Hence $(f+g)h = I_G$ for some $h \in Aut(G)$.
Set $f' = fh$ and $g' = gh$.
Then $f' + g' = I_G$.
Note $(g'+f')(a)$
$= g'(a)f'(a)$
$= (f'(a^{-1})g'(a^{-1}))^{-1}$
$= ((f'+g')(a^{-1}))^{-1}$
$= a$
for all $a \in G$
Thus $f'+g' = g'+f'$
Also $f'(f'+g')$
$= f' \cdot I_G$
$= I_G \cdot f'$
$= (g'+f')f'$
By canceling f'^2,we have $f'g' = g'f'$
By binomial theorem, $(f'+g')^n = \sum_{k=0}^{n} \binom{n}{k} f'^k g'^{n-k}$
Note $f', g' \in Aut(G)$,since $f, g \notin Aut(G)$.
Hence f', g' are nilpotent.
Then $(f'+g')^n = 0$ if n large ,a contradiction. $\square$