$Krull - Schmidt\ Theorem$

Suppose that $G$ satisfies both $ACCN$,$DCCN$, and

$$G = G_1 \times G_2 \times \cdots \times G_s = H_1 \times H_2 \times \cdots \times H_t$$

, where $G_i, H_i$ are indecomposable.
Then (1)$s = t$ (2)there exists $\sigma \epsilon S_t$ such that for each $r \leq t$,

$$G = G_{\sigma(1)} \times G_{\sigma(2)} \times \cdots \times G_{\sigma(r)} \times H_{r+1} \times H_{r+2} \times \cdots \times H_t$$

and $G_{\sigma(i)} \cong H_i$ for $i \leq r$.

ex: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 (= \{(a,b,c)|a,b,c\epsilon\mathbb{Z}_2\})$
$=< (0,1,1) > \times < (1,0,1) > \times < (1,1,0) >$
$=< (1,0,0) > \times < (0,1,0) > \times < (0,0,1) >$
$=< (1,0,1) > \times < (0,1,0) > \times < (0,0,1) >$
$=< (1,0,1) > \times < (1,1,0) > \times < (0,0,1) >$
$=< (1,0,1) > \times < (1,1,0) > \times < (0,1,1) >$

Prove:Suppose that there exists $1 \leq r \leq min(s,t)$ and an injection

$$\sigma : \{1, 2, \cdots, r-1\} \rightarrow \{1, 2, \cdots, s\}$$

such that

$$G = G_{\sigma(1)} \times G_{\sigma(2)} \times \cdots \times G_{\sigma(r-1)} \times H_r \times H_{r+1} \times \cdots \times H_t$$

and $G_{\sigma(i)} \cong H_i$ for $i \leq r-1$.

For example, $r = 1$ then $G = H_1 \times H_2 \times \cdots \times H_t$.

Reordering $G_i$ s.t. $G = G_1' \times G_2' \times \cdots \times G_s'$ where $G_i' = G_{\sigma(i)}$ for $i \leq r-1$.

Write $G = G_1'' \times G_2'' \times \cdots \times G_{r-1}'' \times \cdots \times G_t''$ where $G_i'' = G_i' = G_{\sigma(i)}$ if $i \leq r-1$ and $G_j'' = H_j$ if $j \geq r$.

Let $\Pi_i' = G \rightarrow G_i' \subseteq G$ be the projections. ( viewed as in $End(G)$ )

Note

$$\Pi_r^{''} \cdot 1_G = \Pi_r^{''}(\Pi_1^{'} + \Pi_2^{'} + \cdots + \Pi_s^{'}) = 0 + 0 + \cdots + \Pi_r^{''}\Pi_r^{'} + \Pi_r^{''}\Pi_{r+1}^{'} + \cdots + \Pi_r^{''}\Pi_s^{'}$$

Since $\Pi_r^{''} \restriction_1 H_r = I_{H_r}$ is not a nilpotent in $End(H_r)$, $\Pi_r^{''}\Pi_j^{'} \restriction_{H_r}$ is not nilpotent for some $j \geq r$.

Note $\Pi_r^{''}\Pi_j^{'} \restriction_{H_r} \epsilon \ Aut(H_r)$.

Hence $\Pi_j^{'} \restriction_{H_r}: H_r \to G_j^{'}$ is one to one and $\Pi_r^{''} \restriction_{G_j^{'}}: G_j^{'} \to H_r$ is onto.

Note for $n \geq 1$, $\Pi_r^{''}(\Pi_j^{'}\Pi_r^{''})^n\Pi_j^{'} \restriction_{H_r} = (\Pi_r^{''}\Pi_j^{'})^{n+1} \restriction_{H_r} \epsilon Aut(H_r)$.

Hence $\Pi_j^{'}\Pi_r^{''} \restriction_{G_j^{'}} \epsilon End(G_j^{'})$ is not nilpotent.

Thus $\Pi_j^{'}\Pi_r^{''} \restriction_{G_j^{'}} \epsilon Aut(G_j^{'})$

Hence $\Pi_r^{''} \restriction_{G_j^{'}}: G_j^{'} \to H_r$ is one to one and $\Pi_j^{'} \restriction_{H_r}: H_r \to G_j^{'}$ is onto.

We show $H_j \cong G_j^{'} = G_{u_j}$ for some

$$u_j \epsilon \{1, 2, \cdots, s\} - \{\sigma(1), \sigma(2), \cdots, \sigma(r-1)\}.$$

Extend the definition of $\sigma$ to $\{1, 2, \cdots, r\}$ with $\sigma(r) = u_j$.Then $H_r \cong G_{\sigma(r)}$.

claim: $(G_{\sigma(1)} \times G_{\sigma(2)} \times \cdots \times G_{\sigma(r-1)} \times H_{r+1} \times \cdots \times H_t) \cap G_{\sigma(r)} = \{e\}$

If $a\epsilon(G_{\sigma(1)} \times G_{\sigma(2)} \times \cdots \times G_{\sigma(r-1)} \times H_{r+1} \times \cdots \times H_t) \cap G_{\sigma(r)}$, then $\Pi_r^{''}(a) = e$ and $a = e$ since $\Pi_r^{''} \restriction_{G_{\sigma(r)}}: G_{\sigma(r)} \to H_r$ is isomorphism.

Set $G^* = G_{\sigma(1)} \times G_{\sigma(2)} \times \cdots \times G_{\sigma(r)} \times H_{r+1} \times \cdots \times H_t$.

claim : $G^* = G$

Define a function $\theta : G \to G$ by $\theta = \Pi_1^{''} + \Pi_2^{''} + \cdots + \Pi_{r-1}^{''} + \Pi_j^{'}\Pi_r^{''} + \Pi_{r+1}^{''} + \cdots + \Pi_t^{''}$.

Note:

(1)$\theta \epsilon End(G)$.

(2)$Img(G) = G^*$.

(3)$\theta$ is one to one.

(4)$\theta \epsilon Aut(G)$.

(5)$G = G^*$.

To the end,

if $s \leq t$,

$$G = G_{\sigma(1)} \times G_{\sigma(2)} \times \cdots \times G_{\sigma(s)} \times H_{s+1} \times \cdots \times H_t = G_{\sigma(1)} \times G_{\sigma(2)} \times \cdots \times G_{\sigma(s)}$$

and $s = t$.

if $t \leq s$,

$$G = G_{\sigma(1)} \times G_{\sigma(2)} \times \cdots \times G_{\sigma(t)} = G_1 \times G_2 \times \cdots \times G_s$$

and $s = t$. $\square$