2.4 The action of group on a set

Definition. *An action of a group G on a set is a map $G \times S \to S$ by sending (g, s) into g,s s.t*
1 $e \cdot s = s$ and
2 $(gh) \cdot s = g \cdot (h \cdot s)$ for all $g, h \in G$ and $s \in S$

Note: Let G act on a set S. Then there is a homomorphism from G into the permutation subgroup on S.
i.e $|S| = n \Rightarrow G \to S_n$ a homomorphism.

ex. *$G = S_n$, $S = \{1, 2, ..., n\}$. Then we can define the action of G on S by the usual mapping of a function. i.e $\sigma(i) \in S$ for $\sigma \in G$ and $i \in S$*

ex. *$G < H$. Then*
1. G acts on H by "left translation" i.e $g \cdot h = gh$ for $g \in G$ and $h \in H$
2. H acts on the left cosets of G (by left translations). i.e. $S = \{aG | a \in H\}$ and $H \cdot aG = haG$ for all $h, a \in H$
3. H acts on the right cosets of G by right translations. i.e $S = \{Ga | a \in H\}$ and $h \cdot Ga = Gah^{-1}$ (neither Gha nor Gah, check the definition)

Definition. *Let G act on a set S. For $s \in S$, $O_s := \{g \cdot s | g \in G\}$ is called the orbit of s and $G_s := \{g \in G | g \cdot s = s\}$ is called the stabilizer or isotropy group of s*

ex. *Suppose $G < H$ and G acts on H by left translations. Then for $s \in H$, $O_s = \{gs | g \in G\}$ is a right coset of G, and $G_s = \{e\}$*

Note:
(1) $O_s = O_{s'}$ or $O_s \cap O_{s'} = \phi$ for $s, s' \in S$
(2) $|O_s| = \frac{|G|}{|G_s|}$
(3) If $S = O_{s_1} \cup O_{s_2} \cup ... \cup O_{s_t}$
then $|s| = \sum_{i=1}^{t} |O_{s_i}| = \sum_{i=1}^{t} \frac{|G|}{|G_{s_i}|} =$ the number of $\{s_i | G_{s_i} = G\} + \sum_{G_{s_i} \neq G} \frac{|G|}{|G_{s_i}|}$
(class equation)

ex. *Let p be the smallest prime dividing $|G|$ and $H < G$ with $[G : H] = p$. Then $H \lhd G$.*
Pf: Set $S = \{aH|a \in G\}$. Let G act on S by left translation. This action gives a homomorphism $\phi : G \rightarrow S_p$ where $p = |S| = \frac{|G|}{|H|}$. Let $k = ker\phi$. For $k \in K$,$k \cdot aH = aH$ for any $a \in G$. In particular, by choose $a = e$, we find $k \subseteq H$. Hence $p = \frac{|G|}{|H|} | \frac{|G|}{|K|} = |Img\phi|||s_p| = p!$. Since for each prime $g < p$,$g \nmid |G|$ and then $g \nmid \frac{|G|}{|k|}$, we have $\frac{|G|}{|K|} = p$ and then $H + K$, $K = ker\phi$ is clear to be normal in G.