2.6 Classification of Finite Groups: Part II

(1). $|G| = 1 \;\Rightarrow\; G = \langle e \rangle$.

(2).
$$\begin{aligned} |G| &= p \ (p = 2, 3, 5, 7, 11, 13, \ldots \text{ is a prime}) \\ \Rightarrow G &= \langle a \rangle \ \ (\text{for } e \neq a \in G) \\ &\cong \mathbb{Z}_p \end{aligned}$$

(3).
$$\begin{aligned} |G| &= pq \ (q \nmid p-1) \ (|G| = 15, 33, \ldots) \\ \Rightarrow G &\cong \mathbb{Z}_{pq}. \end{aligned}$$

(4). $|G| = pq \ (q|p-1) \ (|G| = 6, 10, 14, 21, 22, \ldots)$

$\Rightarrow$ (a) abelian

$G \cong \mathbb{Z}_p \times \mathbb{Z}_q. \cong \mathbb{Z}_{pq}.$

(b) nonabelian

$G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q.$

(5) $|G| = p^2 \ (|G| = 4, 9, 25, \ldots)$

$\Rightarrow$ G is abelian

$\Rightarrow$ $G \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$.

(6) $|G| = 8$

$\Rightarrow$ (a) (abelian)

$G \cong \mathbb{Z}_8, \ \mathbb{Z}_4 \times \mathbb{Z}_2, \ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

(b) (nonabelian)

Case $b.1$: $\exists a \in G$ with $|a| = 8$, which contradicts to nonabelian.

Case $b.2$: $\forall e \neq a \in G$ with $|a| = 2$.

$e = (ba)^2 \Rightarrow ba = a^{-1}b^{-1}$

$aba^{-1}b^{-1} = abba = e,$ which contradicts to nonabelian.

Case $b.3$: $a \in G$ with $|a| = 4$

$[G : \langle a \rangle] = 2 \Rightarrow \ \langle a \rangle \ \lhd \ G$

Pick $b \in G \setminus \langle a \rangle$.

Then $G = \langle a \rangle \,\dot{\bigcup}\, b \langle a \rangle$.

Case $b.3$ (i) : $|b| = 2$

$bab^{-1} = a^3 = a^{-1}$ $(|bab^{-1}| = |a|)$
$G \cong < a > \rtimes < b > \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_2 \cong D_4$.
Case $b.3$ (ii) : $|b| = 4$
$|G/ < a > | = 2$
$\Rightarrow b^2 < a > = < a >$
$\Rightarrow b^2 \in < a >$
Since $|b| = 4,\ b^2 = a^2$.
$G = \{b^j a^i | 0 \leq i \leq 3,\ 0 \leq j \leq 1\} \cong Q_8$
with identification
$a \to i,\ b \to j,\ (ab)^2 = a^2 = b^2 \to -1,\ ab \to k$.

(7) $|G| = 12 = 2^2 \times 3$

$\Rightarrow$ (a) (abelian)
$G \cong \mathbb{Z}_4 \times \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$

$\Rightarrow$ (b) (nonabelian)
Pick $< a > \in \text{Syl}_3(G)$
Let G act on left coset of $< a >$ in G.
This gives a homomorphism $\phi : G \to S_4$ (since $[G : < a >] = 4$)
and Ker$\phi \subseteq < a >$.
Hence ker$\phi = < e >$ or $< a >$ (since $| < a > | = 3$)
If ker$\phi = < e >$,
then $G \cong \phi(G) = A_4$ by checking all subgroups of order 12 in S_4.
(In fact, A_4 is the only subgroup of S_4 of index 2 and $|S_4|/2 = |A_4| = 12$.)
Suppose ker$\phi = < a >$.
Then $|G/ < a > | = 4$ and $G/ < a >$ is abelian.
(In fact, any group with four elements is abelian.)
Hence $G/ < a > \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.
Case 1: $G/ < a > \cong \mathbb{Z}_4$.
$G = < a > \ \dot{\bigcup}\ b < a > \ \dot{\bigcup}\ b^2 < a > \ \dot{\bigcup}\ b^3 < a >$, where $b^4 \in < a >$ $(*)$.
If $b^4 = a, a^{-1}$,
then $|b| = 12$, which contradicts to the fact that G is not abelian.
Then $b^4 = e$
Since $< a > \ \lhd\ G,\ bab^{-1} = a^{-1}$.
(If $bab^{-1} = a$ then G is abelian by $(*)$)
Hence $G \cong\ < a > \rtimes < b >$.
Case 2: $G/ < a > \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
Then $G = < a > \ \dot{\bigcup}\ b < a > \ \dot{\bigcup}\ c < a > \ \dot{\bigcup}\ bc < a >$
where $b^2,\ c^2,\ (bc)^2 \in < a >$.

If $b^2 = a,\ a^{-1}$ then $|b| = 6$.
If $b^2 = e$ and $ab = ba$ then $|ab| = 6$.
If $b^2 = e$ and $ab \neq ba$ then $bab^{-1} = a^{-1}$.
Hence there exists $t \in G$ with $|t| = 6$
unless $bab^{-1} = a^{-1},\ cac^{-1} = a^{-1},\ (bc)a(bc)^{-1} = a^{-1}$ and $b^2 = c^2 = (bc)^2 = e$.
In this case, $a^{-1} = (bc)a(bc)^{-1} = b(cac^{-}) = ba^{-1}b^{-1} = (bab^{-1})^{-1} = a,$
a contradiction.
Since $[G :< t >] = 2,\ \ < t > \ \lhd\ G$.
Pick $u \in G \setminus < t >$.
Then $G =< t > \ \dot{\bigcup}\ u < t >,$ where $u^2 \in< t >$.
If $u^2 = t$ or t^{-1} then $G =< u >$ is abelianm which is a contradiction.
If $u^2 = t^3$ then $u^4 = t^6 = e$ and we have
$G/ < a >=< u < a >>\cong \mathbb{Z}_4,$ which is a contradiction.
If $u^2 = t^2$ or $t^4,$ then $|u| = 6$ and $< u > \ \lhd G$
Assume $u^2 = t^2$. Note that $utu^{-1}t^{-1} \neq e$ since G is not abelian.
Hence $utu^{-1} = t^{-1}$ subce $|utu^{-1}| = |t^{-1}|$.
and similarly $tut^{-1} = u^{-1}$
Then $u^2 = u(tu^{-1}t^{-1}) = t^{-2},$ which is a cntradiction.
Simiilar for $u^2 = t^4,$ we have another contradiction.
Then $u^2 = e$ and $utu^{-1} = t^{-1}$
Hence $G \cong< t > \rtimes < u >$.
With $t \to \sigma,\ u \to \tau,$ we have $G \cong D_6$