2.8 Normal and subnormal series

Definition 1. *$< e >= G_n \lhd G_{n-1} \lhd G_{n-2} \lhd \cdots \lhd G_1 \lhd G_0 = G$ is called a* **subnormal series***.*

Warning: It is not necessary $G_i \lhd G$.

Definition 2. *A subnormal series is* **solvable** *if $G_n =< e >$ and G_i/G_{i+1} is abelian.*

Theorem 1. *G is solvable $\Leftrightarrow$ G has a solvable subnormal series.*

Proof. ($\Rightarrow$) Set $G_i = G^{(i)}$.Then $< e >= G_n \lhd G_{n-1} \lhd \cdots \lhd G_0 = G$ and G_i/G_{i+1} is abelian.

($\Leftarrow$) Suppose G has a solvable subnormal series, $< e >= G_n \lhd G_{n-1} \lhd \cdots \lhd G_0 = G$.
Since G_i/G_{i+1} is abelian, $G_{i+1} \geq G_i^{'} \geq (G_{i-1}^{'})^{'} = G_{i-1}^{(2)} \cdots \geq G^{(i+1)}$.
Hence $< e >= G_n \geq G^{(n)}$ for some n.

□