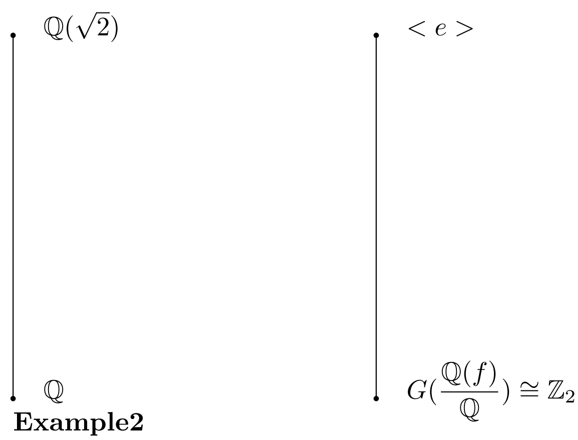
5.4 The Galois Group of a polynomial

Definition 1. *Let* $f(x) \ \epsilon \ K[x]$*have roots of* $\alpha_1, \alpha_2, ..., \alpha_k$ *with* $k = degf$*.*
Then $K(f) := K(\alpha_1, \alpha_2, ..., \alpha_K)$ *and* $G(\frac{K(F)}{K})$ *is the Galois Group of* $f(x)$ *over* K*.*

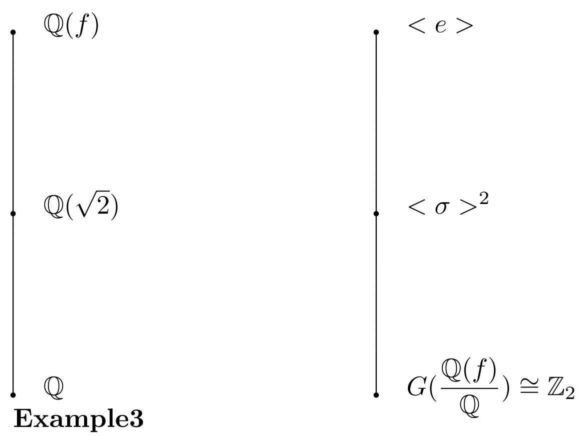
Example1
$f(x) = x^2 - 2 \ \epsilon \ \mathbb{Q}[x]$, $\mathbb{Q}(f) = \mathbb{Q}(\sqrt{2})$



Example2
$f(x) = x^4 + 4x^2 + 2 \ \epsilon \ \mathbb{Q}[x]$
$f(x) = 0, x^2 = \frac{-4 \pm \sqrt{8}}{2} = -2 \pm \sqrt{2}$
$\Rightarrow x = \pm\sqrt{-2 \pm \sqrt{2}}$
$\mathbb{Q}(f) = \mathbb{Q}(\sqrt{-2 + \sqrt{2}})$, where $\sqrt{2} \ \epsilon \ \mathbb{Q}(\sqrt{-2 + \sqrt{2}})$

$$\sigma \ \epsilon \ G(\frac{\mathbb{Q}(f)}{\mathbb{Q}}) \Rightarrow \sigma(\sqrt{-2 + \sqrt{2}}) = \pm\sqrt{-2 \pm \sqrt{2}}$$

Say $\sigma(\sqrt{-2 + \sqrt{2}}) = \sqrt{-2 - \sqrt{2}}$
Then $\sigma(\sqrt{(-2 + \sqrt{2})^2}) = \sigma(-2 + \sqrt{2}) = -2 + \sigma(\sqrt{2}) = -2 - \sqrt{2}$
$\sigma(\sqrt{-2 - \sqrt{2}}) = \frac{\sigma(\sqrt{2})}{\sigma(\sqrt{-2 + \sqrt{2}})} = \frac{-\sqrt{2}}{\sqrt{-2 - \sqrt{2}}} = -\sqrt{-2 + \sqrt{2}}$
Hence, $\sigma^2(\sqrt{-2 + \sqrt{2}}) = -\sqrt{-2 + \sqrt{2}}$
Then, $\sigma^4 = 1$
Hence, $G(\frac{\mathbb{Q}(f)}{\mathbb{Q}}) = < \sigma > \cong \mathbb{Z}_4$



Example3

$f(x) = x^2 - 10x^2 + 4$

$f(x) = 0 \Rightarrow x^2 = \frac{10 \pm \sqrt{84}}{2} = 5 \pm \sqrt{21}$

$\Rightarrow x = \pm\sqrt{5 \pm \sqrt{21}}$

Then $\mathbb{Q}(f) = \mathbb{Q}(\sqrt{5+\sqrt{21}})$, where $\frac{2}{\sqrt{5+\sqrt{21}}} = \sqrt{5-\sqrt{21}} \; \epsilon \; \mathbb{Q}(\sqrt{5+\sqrt{21}})$

$\sigma \; \epsilon \; G(\frac{\mathbb{Q}(f)}{\mathbb{Q}}) \Rightarrow \sigma(\sqrt{5+\sqrt{21}}) = \sqrt{5-\sqrt{21}}$

Say $\sigma(\sqrt{5+\sqrt{21}}) = \pm\sqrt{5-\sqrt{21}}$

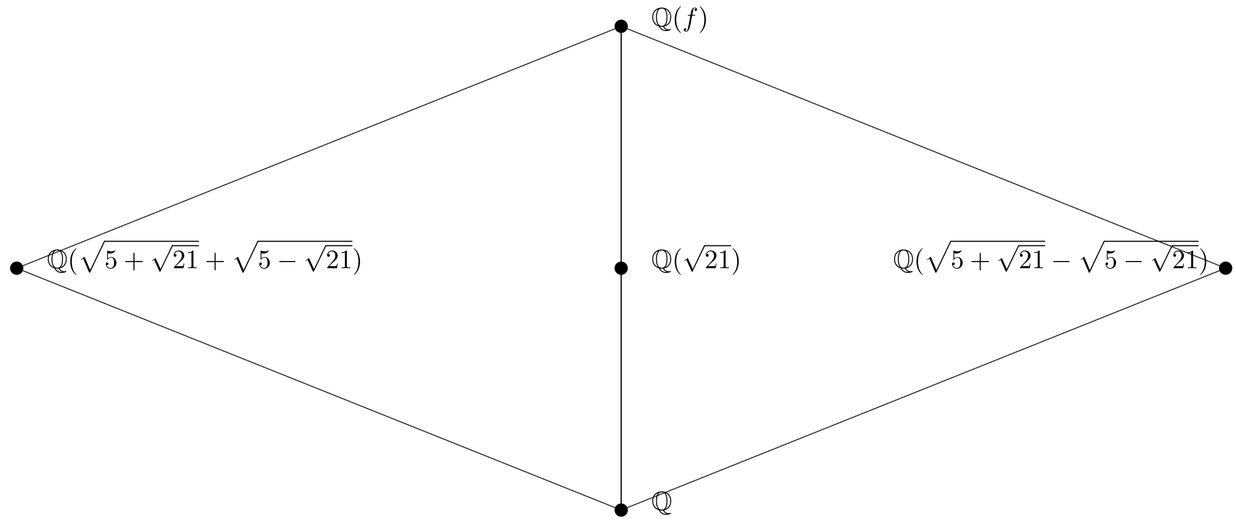
$\sigma(\sqrt{5-\sqrt{21}}) = \frac{\sigma(2)}{\sigma(\sqrt{5+\sqrt{21}})} = \frac{2}{\sqrt{5-\sqrt{21}}} = \sqrt{5+\sqrt{21}}$

Then $\sigma^2 = e$

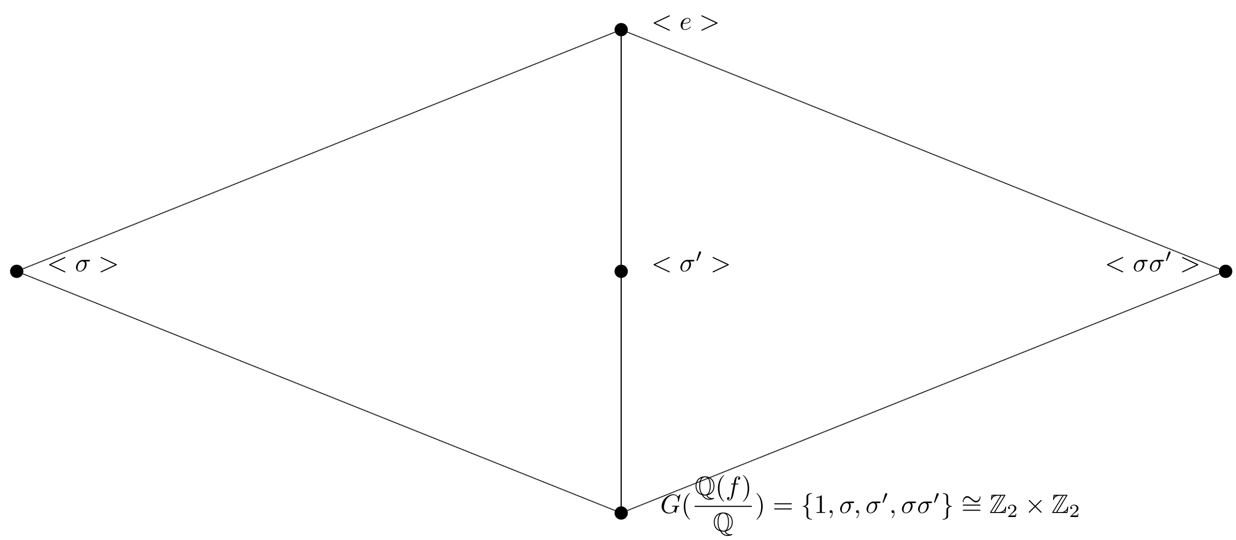
Say$\sigma'(\sqrt{5+\sqrt{21}}) = -\sqrt{5+\sqrt{21}}$

Then $\sigma'^2 = e$

Hence,$G(\frac{\mathbb{Q}(f)}{\mathbb{Q}}) = \{1, \sigma, \sigma', \sigma\sigma'\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$



$\mathbb{Q}(f)$
$\mathbb{Q}(\sqrt{5+\sqrt{21}}+\sqrt{5-\sqrt{21}})$
$\mathbb{Q}(\sqrt{21})$
$\mathbb{Q}(\sqrt{5+\sqrt{21}}-\sqrt{5-\sqrt{21}})$
$\mathbb{Q}$



$< e >$
$< \sigma >$
$< \sigma' >$
$< \sigma\sigma' >$
$G(\frac{\mathbb{Q}(f)}{\mathbb{Q}}) = \{1, \sigma, \sigma', \sigma\sigma'\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$