§ 9. Radical Extension

Recall: $ax^2 + bx + c = 0$ has roots $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

if characteristic of the field is not 2.

Question: Can the roots of $a_n x_n + a_{n-1} x_{n-1} + \ldots +$

$a_1 x + a_0 = 0$

be written as a form with only field operations and $\sqrt[k]{}$

on the elements $a_0, a_1, \ldots, a_n$?

Definition 0.1. An extension field F of K is a radical extension if $F = K(u_1, u_2, \ldots, u_r)$ for some r, where $u_i^{n_i} \in K(u_1, u_2, \ldots, u_{i-1})$ for $1 \leq i \leq r$.

Example 0.2. $\sqrt[7]{5 + \sqrt[4]{6 + \sqrt{2}}} \in Q(\sqrt{2}, \sqrt[4]{6 + \sqrt{2}}, \sqrt[7]{5 + \sqrt[4]{6 + \sqrt{2}}}\ )$

Question: For a polynomial equation $x_n + a_1 x_{n-1} + \ldots + a_{n-1}x + a_0 = 0$, let K be a field containing $a_1, a_2, \ldots, a_n$. Does there a radical extension F that contains all roots of the polynomial equation?

Example 0.3. *$x^2 - 2 = 0$ has roots $\pm\sqrt{2}$*

$\pm\sqrt{2} \in Q(\sqrt{2})$, where $(\sqrt{2})^2 \in Q$. Hence $Q(\sqrt{2})$ is radical extension of Q.