5.9 Radical Extension Part II

Recall:

1. For groups $N \triangleleft G$, G is solvable $\Leftrightarrow$ $N, G/N$ are solvable.

2. Abelian groups are solvable.

Theorem: Let F be a radical extension of K, then $G(F/K)$ is solvable.

sketch of proof:

1. $F' = K(\alpha)$, where α is a primitive nth root of 1 :
$G(F'/K) \cong U_n$ is abelian and hence solvable.

2. $F'' = K(\alpha)(f)$ where $f(x) = x^n - a \in K[x]$ (roots of $f(x)$ are $\sqrt[n]{a}\alpha^i$):
It is immediate to check $G(F''/K(\alpha))$ abelian, then solvable.

3. $G(F''/K(\alpha)) \subseteq G(F''/K)$, $G(F''/k)/G(F''/K(\alpha)) \cong G(K(\alpha)/k)$ is solvable by 1.
By this 2. and **Recall** 2., we have $G(F''/K)$ solvable.
By repeating 1.– 3., we have $G(E/K)$ solvable for some E containing F.

4. $F \subseteq E \Rightarrow G(F/K) \subseteq G(E/K)$, $G(E/K)$ solvable $\Rightarrow G(F/K)$ solvable