Fundamental Theorem of Algebra

Lemma：$[F:C] \neq 2$ for any field F containing C.

pf : Suppose $[F:C]=2$

Then $F = C(\alpha)$ with an irreducible polynomial $f(x) = x^2 + bx + c \in C[x]$ s.t. $f(\alpha)=0$.

But $f(x) = \left(x - \dfrac{-b+\sqrt{b^2-4c}}{2}\right)\cdot\left(x - \dfrac{-b-\sqrt{b^2-4c}}{2}\right)$ is not irreducible.

Theorem (FTA)：If $f(x) \in C[x]$, then $f(x)$ has a root in C.

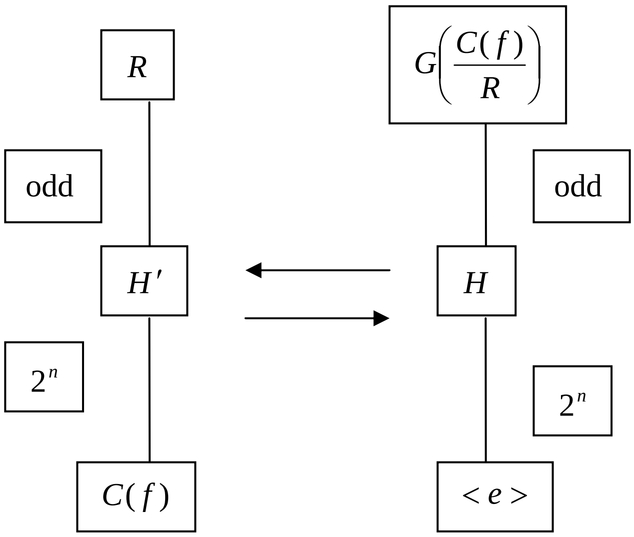
pf : Consider field extension $R \subseteq C \subseteq C(f)$.

We want to prove $C \subseteq C(f)$.

Since $[C(f):R]$ is even, 2 divides $G\left(\dfrac{C(f)}{R}\right)$.

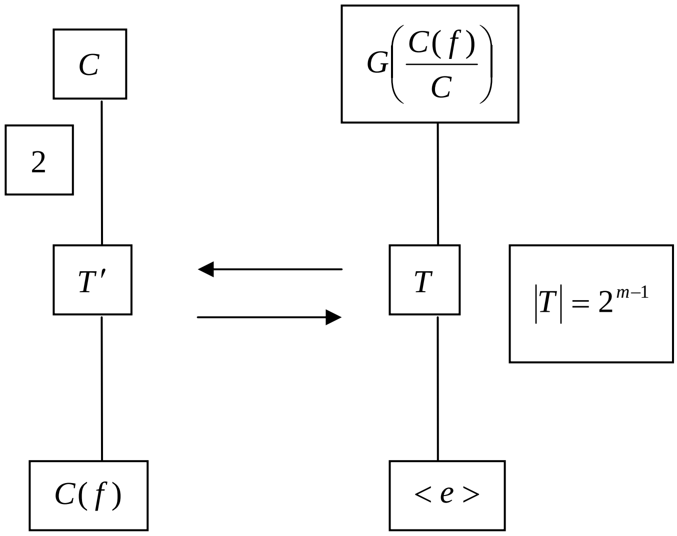
Pick $H \in Syl_2\left(G\left(\dfrac{C(f)}{R}\right)\right)$.

Let H' be the fixed field of H.



Hence $[H':R]$ is odd.

Since any polynomial of odd degree over R has a root in R, on field extension of R with odd dimension.
Hence $H' = R$.
Hence $H = G\left(\frac{C(f)}{R}\right)$.
Then $\left|G\left(\frac{C(f)}{R}\right)\right| = 2^n$ for some $n \in N$.
Note $G\left(\frac{C(f)}{C}\right) \subseteq G\left(\frac{C(f)}{R}\right)$.
Hence $\left|G\left(\frac{C(f)}{C}\right)\right| = 2^m$ for some $1 \le m \le n$.



By Sylow 1st theorem, there exist $T < G\left(\frac{C(f)}{C}\right)$ with $|T| = 2^{m-1}$.
Then $[T' : C] = 2$, a contradiction to previous lemma.