進階代數(下) 第十次作業

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1. Let n be a positive integer and let $U_n := \{m \mid 0 \le m \le n-1,\ (m,n)=1\}$.
   (a) (賴德展) How many elements in U_{100}?
   (b) (劉侖欣) Show that U_n is a group under multiplication modulo n.
   (c) (周彥伶) Let p be a prime. Show that $a^p \equiv a \pmod p$.
   (d) (洪湧昇) Let $q < p$ be primes and $q|p-1$. Suppose that you have learned the group U_p to be cyclic. Show that there exists an integer s such that $s \not\equiv 1$ and $s^q \equiv 1 \pmod p$.

2. Let $q < p$ be two primes and $q|p-1$. Let s be an integer satisfying 1(d).
   (a) (林志峰) Show that the map $\alpha : \mathbb{Z}_p \to \mathbb{Z}_p$ given by $\alpha(i) = si$ is an additive group automorphism.

      Proof. Assume $i + j \equiv t \pmod p$ for $i, j, t \in \mathbb{Z}_p$. Then $\alpha(i + j) = st \equiv si + sj = \alpha(i)+\alpha(j)$. This shows the homomorphism of α. If $\alpha(i) = \alpha(j)$ then $si = sj$. Hence $i = j$. This shows the injective of α. The surjective follows form the injective since $\mathbb{Z}_p$ is finite.
   (b) (黃正一) Show that the map $\theta : \mathbb{Z}_q \to \operatorname{Aut} \mathbb{Z}_p$ given by $\theta(i) = \alpha^i$ is a homomorphism.

      Proof. Assume $i + j \equiv t \pmod q$ for $i, j, t \in \mathbb{Z}_q$. Note that $s^i s^j \equiv s^t \pmod p$ by the choice of s in 1(d). Hence $\alpha^i\alpha^j = \alpha^t$. This shows the homomorphism of α.
   (c) (邱鈺傑) Let $\mathbb{Z}_p \rtimes_\theta \mathbb{Z}_q$ be the semidirect product of $\mathbb{Z}_p$ and $\mathbb{Z}_q$ as defined in Homework 7. Show that $|(1, 0)| = p$, $|(0, 1)| = q$ and $(0, 1)(1, 0) = (s, 0)(0, 1)$. $\mathbb{Z}_p \rtimes_\theta \mathbb{Z}_q$ is called the *metacyclic group*.

      Proof This is immediate from the definition of semidirect product.

3. (蕭雯華) (Normalizer grows in p-group) Let $|G| = p^n$ where p is a prime, $H < G$ and $H \ne G$. Show that $H \ne N_G(H)$.

4. (陳巧玲) Suppose $|G| = p^n$ and $< e > \ne H \lhd G$. Show $H \cap Z(G) \ne < e >$.

5. (林詒琪) Let $|G| = p^n$. Show that for each $0 \le k \le n$, G has a normal subgroup of order p^k.

6. (葉彬) Let H be a normal subgroup of order p^k of a finite group G. Show that H is contained in every Sylow p-subgroup of G.