Solution for Homework 10 part 2, problem 3 to 6

**3.** (Normalizer grows in $p$-group) Let $|G| = p^n$ where $p$ is a prime, $H < G$ and $H \neq G$. Show that $H \neq N_G(H)$.

**Solution:**
By p.94 Corollary5.6 : $H$ is a $p$-subgroup of a finite group $G$ such that $p$ divides $[G : H]$, then $N_G(H) \neq H$.

$|G| = p^n < \infty \Rightarrow$

**(i)** If $n = 1$, $|G| = p$ , by Lagrange Theorem, $|H| = 1$ or $p$ , but $H \neq G$ $\Rightarrow H = \{e\}$. Hence, $N_G(H) = G \neq \{e\} = H$.

**(ii)** If $n > 1$ , $|G| = p^n$, by Lagrange Theorem, $H$ is a $p$-subgroup of $G$. $[G : H] = \frac{|G|}{|H|} = p^t$ for some $t \in \mathbb{N}$. By Corollary 5.6, $N_G(H) \neq H$.

**Teacher:**
$H$ acts left coset of $H \Rightarrow [G : H] = [N_G(H) : H] + \sum\limits_{H_{p_i} \neq H} \frac{|H|}{|H_{p_i}|}$
(since $g \cdot tH = tH, \forall g \in H \Leftrightarrow t^{-1}gt \in H \Leftrightarrow t^{-1}Ht = H \Leftrightarrow t \in N_G(H)$.
For $t_1, t_2 \in N_G(H)$ such that $t_1H = t_2H \Leftrightarrow t_2^{-1}t_1 \in H$)

**4.** Suppose $|G| = p^n$ and $< e > \neq H \triangleleft G$. Show that $H \cap Z(G) \neq < e >$.

**Solution:**
Define a group action of $G$ on $H$, $g * h = ghg^{-1}$, $\forall g \in G$ and $h \in H$.

First, we have to check this is the group action of $G$ on $H$. (since $H \triangleleft G$)
(i)$e * h = ehe^{-1} = h$ , for $h \in H$.
(ii)$(g_1g_2) * h = (g_1g_2)h(g_1g_2)^{-1} = g_1(g_2hg_2^{-1})g_1{-}1 = g_1 * (g_2 * h)$

Second, for $|H| = \sum\limits_{i,h_i \in H} |O_{h_i}| = \sum_{i,h_i \in H} 1 + \sum_{|O_{h_i}| \neq 1} |O_{h_i}|$

$|O_{h_i}| = 1$

$\Leftrightarrow O_{h_i} = \{g * h_i | g * h_i = h_i\}$

$\Leftrightarrow O_{h_i} = \{g * h_i | h_i = g * h_i = gh_ig^{-1}, \forall g \in G, \text{ and } h_i \in H\}$

$\Leftrightarrow O_{h_i} = \{g * h_i | gh_i = h_ig, \forall g \in G\}$

$\Leftrightarrow h_i \in Z(G)$

So, $|H| = \sum\limits_{i,h_i \in Z(G) \cap H} 1 + \sum_{|O_{h_i}| \neq 1} |O_{h_i}|$

Since $H < G$ , by Lagrange Theorem, $|H| \mid |G| = p^n$, so $|H| = p^t$, $1 < t < n$.
This implies, $p \mid |H| = \sum\limits_{h_i \in Z(G) \cap H} 1 + \sum_{|O_{h_i}| \neq 1} |O_{h_i}| = \sum_{h_i \in Z(G) \cap H} 1 + \sum_{|O_{h_i}| \neq 1} \frac{|G|}{|G_{h_i}|}$
, where$|G| = p^n$, $|G_{h_i}| = p^{\alpha_n}$.
So , $p \mid \sum\limits_{h_i \in Z(G) \cap H} 1$
Thus, $H \cap Z(G) \neq < e >$

**5.** Let $|G| = p^n$. Show that for each $0 \leq k \leq n$, $G$ has a normal subgroup of order $p^k$.

**Solution:**
$n = 1$ , it is trivial.

Suppose it is true for $|G| = p^i$, $i < n$
Let $G$ act on $G$ by conjugation.
$|G| = |Z(G)| + \sum\limits_{i, G_{\rho_i} \neq G} \frac{|G|}{|G_{\rho_i}|}$ , $p \mid Z(G)$
By Cauchy Theorem, $\exists a \in Z(G)$, $|a| = p$ , $< a > \lhd G$
$\varphi : G \to G/ < a >$, $|G/ < a > | = p^{n-1}$ , by induction hypothesis,
$\forall k$ , $0 \leq k \leq n - 1$ , $\exists F$ , $F \lhd G/ < a >$ , $|F| = p^k$

Claim: $\varphi^{-1}(F) \lhd G$ , i,e, $a\varphi^{-1}(F)a^{-1} \subseteq \varphi^{-1}(F)$
Take $h \in \varphi^{-1}(F)$ , $\varphi(aha^{-1}) = \varphi(a)\varphi(h)(\varphi(a))^{-1} \in F$ , $\Rightarrow aha^{-1} \in \varphi^{-1}(F)$

$\varphi^{-1}(F) \lhd G$ , $|\varphi^{-1}(F)| = p^k p = p^{k+1} = p^s$ , $1 \leq s \leq n$
When $s = 0$ , $\{e\} \lhd G$.

**6.** Let $H$ be a normal subgroup of order $p^k$ of a finite group $G$. Show that $H$ is contained in every Sylow $p$-subgroup of $G$.

**Solution:**
For any $G' \in Sylp(G)$ , $S = \{gG' | g \in G\}$ , $h \circ S = hgG'$
$p \nmid \frac{|G|}{|G'|} = \sum\limits_{|O_{\rho_i}| = 1} 1 + \sum\limits_{|O_{\rho_i}| \neq 1} \frac{|H|}{|Hg_i|}$ $(|H| = p^k)$
$\exists g_i \in G$ such that $hg_iG' = g_iG'$ , $\forall h \in H \Rightarrow g_i^{-1}hg_iG' = G'$ , $\forall h \in H$
$\therefore g_i^{-1}hg_i \in G'$ , $\forall h \in H \Rightarrow g_i^{-1}Hg_i \subseteq G'$.
$\because H$ is normal $\therefore g_i^{-1}Hg_i = H \Rightarrow H \subseteq G'$.