Homework 11

1.$H \in Sylp(G)$
$H \lhd G$
$H', H \in Sylp(G)$
$H' = g^{-1}Hg = H$ for some $g \in G$
$Sylp(G) = H$

2.$|Sylp(G)| = kp + 1$ for some k
$|Sylp(G)|\ ||G|$
$\Rightarrow kp + 1|p^n q$
$|Sylp(G)| = 1$
say $H \in Sylp(G)$
claim:$\forall g, g^{-1}Hg \in H$
$|g^{-1}Hg| = |H|$
$g^{-1}Hg \in Sylp(G)$
by uniqueness we have $g^{-1}Hg = H$

3.(1)$|G| = 12 = 2^2 \cdot 3$
$|Syl_3(G)| = 3k + 1|4 \Rightarrow |Syl_3(G)| = 1$ or 4
Case 1: $|Syl_3(G)| = 1$, say $H \in Syl_3(G) \Rightarrow H \lhd G$
Case 2: $|Syl_3(G)| = 4$
Note: If $P, Q \in Syl_3(G)\ and\ P \neq Q \Rightarrow P \cap Q =< e >$
$\exists 4(3 - 1) = 8$ elements
The remaining $12 - 8 = 4$ elements from a unique 2-Sylow subgroup K
$\Rightarrow K \lhd G$
$\therefore$ it is not simple.
(2)$|G| = 28 = 2^2 \cdot 7$
$|Syl_7(G)| = 7k + 1 \mid 4 \Rightarrow |Syl_7(G)| = 1$
Say $H \in Syl_7(G) \Rightarrow H \lhd G \Rightarrow$ it is not simple.
(3)$|G| = 56 = 2^3 \cdot 7$
$|Syl_7(G)| = 7k + 1|8 \Rightarrow |Syl_7(G)| = 1$ or 8
Case 1: $|Syl_7(G)| = 1$, say $H \in Syl_7(G) \Rightarrow H \lhd G$
Case 2: $|Syl_7(G)| = 8$
Note: If $P, Q \in Syl_7(G)\ and\ P \neq Q \Rightarrow P \cap Q =< e >$
$\exists 8(7 - 1) = 48$ elements
The remaining $56 - 48 = 8$ elements from a unique 2-Sylow subgroup K
$\Rightarrow K \lhd G$
$\therefore$ it is not simple.
(4)$|G| = 200 = 2^3 \cdot 5^2$
$|Syl_5(G)| = 5k + 1 \mid 8 \Rightarrow |Syl_5(G)| = 1$
Say $H \in Syl_5(G) \Rightarrow H \lhd G \Rightarrow$ it is not simple.

4.Let G be a simple group of order 168. $\therefore |G| = 2^3 \cdot 2 \cdot 7$
Now we consider the number of Sylow 7-subgroups
By Sylow's Third Theorem
$|Syl_7(G)| \equiv 1$ mod 7 and $|Syl_7(G)|\ |24$
Hence $|Syl_7(G)| \equiv 1$ or 8
$\because G$ is a simple group $\therefore |Syl(G)| = 8$
Thus, there are 8 distinct cyclic subgroup H_i in G of order 7 for $i \in \{1, 2, \cdots, 8\}$
$\because H_i \cap H_j = \{e\}$ for any $i, j \in \{1, 2, \cdots, 8\}$ and $i \neq j$
$\therefore$ There are 48 elements of order 7 in G.