進階代數(下) 第十二次作業

上課老師: 翁志文

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1. (a) (洪湧昇) Suppose $|G| = p^n$ with p a prime. Show that for each $1 \leq i \leq n$ there exists a subgroup of order p^i.

   (b) (林志峰) Suppose $|G| = p^n$. Show that G is nilpotent.

   (c) (黃正一) Show that the direct product of a finite number of nilpotent groups is nilpotent.

   (d) (邱鈺傑) Let H be a proper subgroup of a nilpotent group G. Show that H is a proper subgroup of its normalizer $N_G(H)$.

   (e) (蕭雯華) Suppose G is nilpotent. Show that G is the direct product of its Sylow subgroups.

   (f) (陳巧玲) Suppose G is finite nilpotent and m divides $|G|$. Show that G has a subgroup of order m.

2. Set

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

   Let S_i denote the lit-only move on P_3 associated with $i \in \{1, 2, 3\}$, interpreted as a 3×3 matrix over $\mathbb{Z}_2$. Let $\mathbf{L}$ be the group generated by lit-only moves in P_3, and let $\mathbf{L}$ act on $\mathbb{Z}_2^3$ (set of column vectors) by left multiplication with the set $\mathcal{O}$ of orbits. Let $\mathbf{L}^t := \{S^t \mid S \in \mathbf{L}\}$, and let $\mathbf{L}^t$ act on $\mathbb{Z}_2^3$ by left multiplication with set $\mathcal{R}$ or orbits.

   (a) (林詒琪) Find S_iA, AS_i^t, AS_i and S_i^tA for $1 \leq i \leq 3$.

   (b) (葉彬) Find $\mathcal{O}$.

   (c) (林育生) Find $\mathcal{R}$.